SISMID Module 6: Stochastic Epidemic Models with Inference – Exercise Session 1

Dongni Zhang, Stockholm University

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Exercise 1.1

Final Size of Outbreak

Final Size Equation

$$1 - \tau = e^{-R_0 \tau}.$$

Note: This equation only gives the final fraction infected among the initially susceptible individuals.

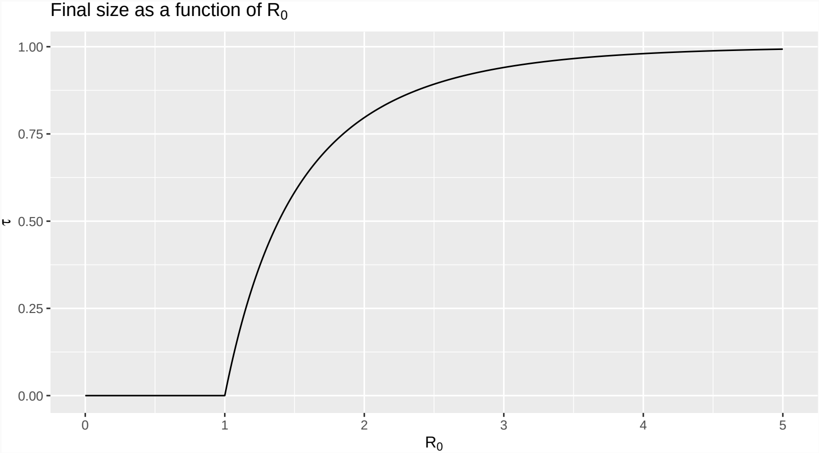
- There is always a solution $\tau = 0$.
- If $R_0 > 1$, there exists a second solution $\tau^* > 0$.
- Final size τ shall be the largest solution on $[0, 1]$.

Plot of final size as a function of R_0

Procedure in R:

- Set a function of R_0 solving for τ numerically, return $\tau = \tau(R_0)$.
- Create a vector of 10000 R_0 values in $[0, 5]$.
- Create a vector of corresponding values of τ.
- Plot τ against R_0.

Plot of final size as a function of R_0



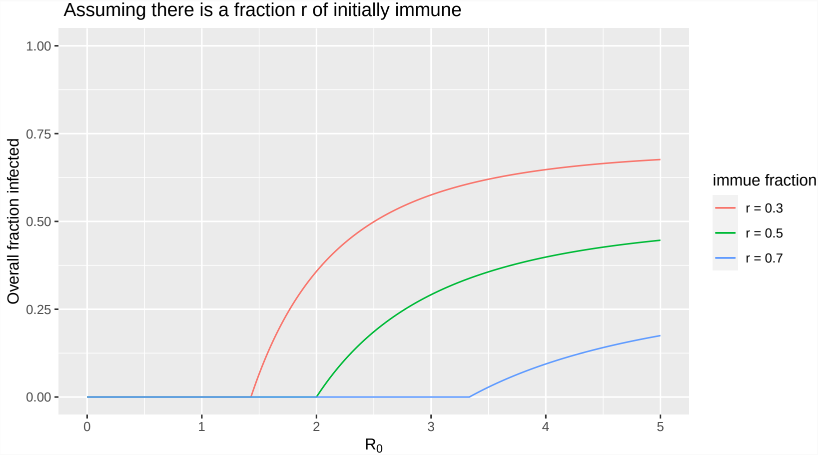
Now assuming initial fraction of immune

If there is a fraction r of initially immunes, then there is fraction $(1-r)$ of initially susceptibles. Then the final size among initially susceptibles τ^* solves

$$1-\tau = e^{-R_0(1-r)\tau}.$$

Then the **overall** fraction infected shall be $\tau^*(1-r)$.

Plot of overall fraction infected



Exercise 1.2

Markovian SIR Epidemic Model

- Consider a closed and homogeneous mixing population with fixed size N.
- At any time point, each individual is susceptible, infectious or recovered.
- The rate of infectious contacts is β, so the rate at which one infectious has contact with a specific other individual is β/N.
- Once infected, one remains to be infectious for a period $I \sim \mathrm{Exp}(\gamma)$, after which one becomes recovered and immune.

Markovian SIR Epidemic Model

Let $S(t), I(t), R(t)$ be the number of individuals in different states S, I, R at time t respectively.

Two types of events:

- $S \to I$: a susceptible gets infected.
- $I \to R$: a infectious individual recovers.

The corresponding rates:

- $\frac{\beta}{N} S(t) I(t)$
- $\gamma I(t)$

Note: at any time t, $S(t) + I(t) + R(t) = N$.

Deterministic SIR Epidemic Model

SIR differential equation system

$$\frac{dS(t)}{dt} = -\frac{\beta}{N}S(t)I(t),$$
$$\frac{dI(t)}{dt} = \frac{\beta}{N}S(t)I(t) - \gamma I(t).$$

Initial Conditions

$$S(0) = N - 1, I(0) = 1.$$

Numerical Solution of the SIR ODE

- Define the function to compute derivatives $(dS(t)/dt, dI(t)/dt)$ for the SIR ODE.

```
gamma <- 0.25
beta <- 0.75
deter$\_$sir <- function(t,y, parms) {
  beta <- parms[1]
  gamma <- parms[2]
  N <- parms[3]
  S <- y[1]
  I <- y[2]
  return(list(c(S=....,
                I=....
                )))
}
```

Numerical Solution of the SIR ODE

- Solve the SIR differential equation system with initial conditions (Use deSolve::lsoda):

```
lsoda(y= ..., \#initial conditions
times= ...., \#times at which explicit estimates for y are desired
func= ..., \#an R-function that computes the values of derivatives in the ODE
parms= ... \#vector or list of parameters used in func
)
```

Stochastic SIR Epidemic Model

Described as a continuous-time Markov process:

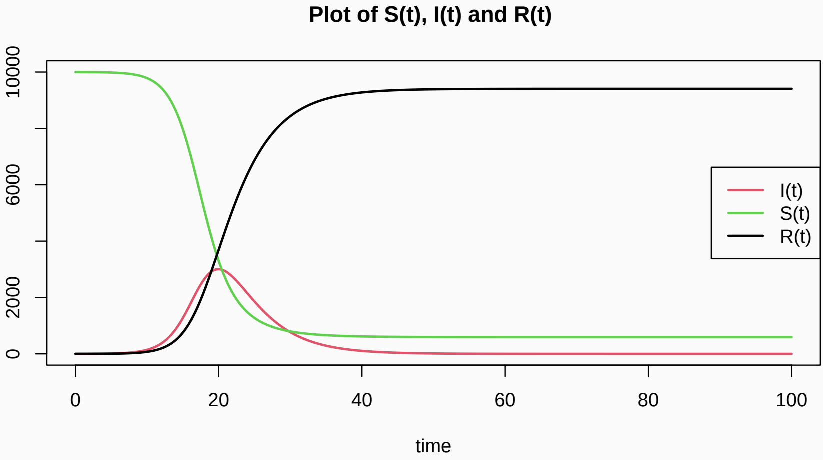
Events	Transition	Rates
Infection	$(S(t), I(t)) \to (S(t)-1, I(t)+1)$	$\frac{\beta}{N} S(t) I(t)$
Recovery	$(S(t), I(t)) \to (S(t), I(t)-1)$	$\gamma I(t)$

Once $I(t) = 0$, the epidemic stops.

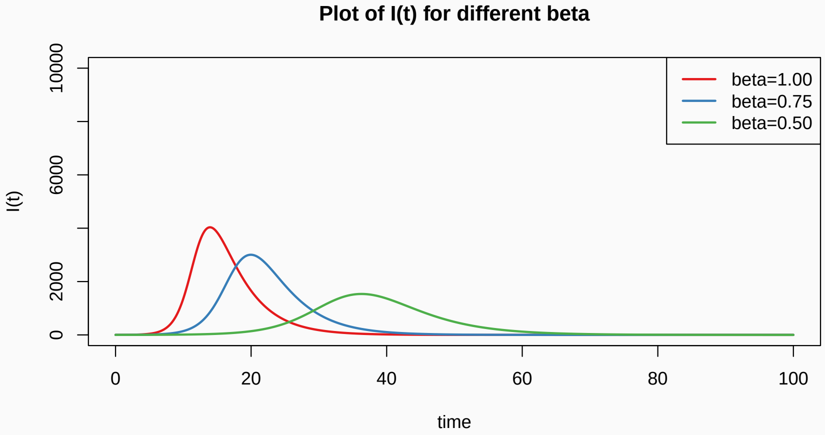
Algorithm to decide which event occurs first:

- From those two rates, we draw two exponential random numbers for each possible event. `rexp(...)`
- Determine the event with the smaller random number. `which.min(...)`
- Record the event time and update the number of S and I according to the event type.

Plot the deterministic curves of $S(t), I(t)$ and $R(t)$

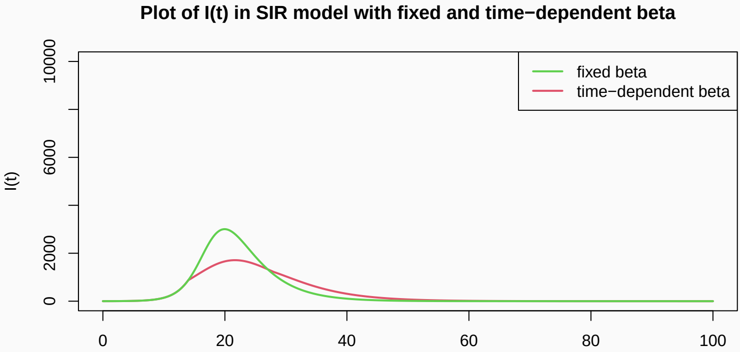


Plot of $I(t)$ with different β

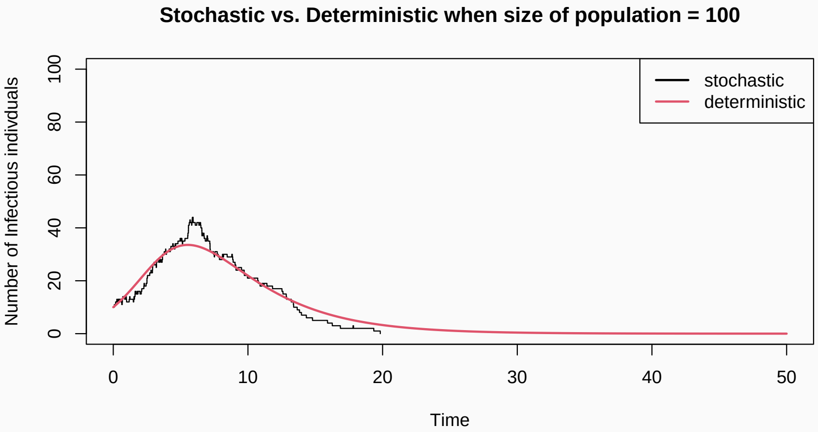


Plot of $I(t)$ when β is time-dependent

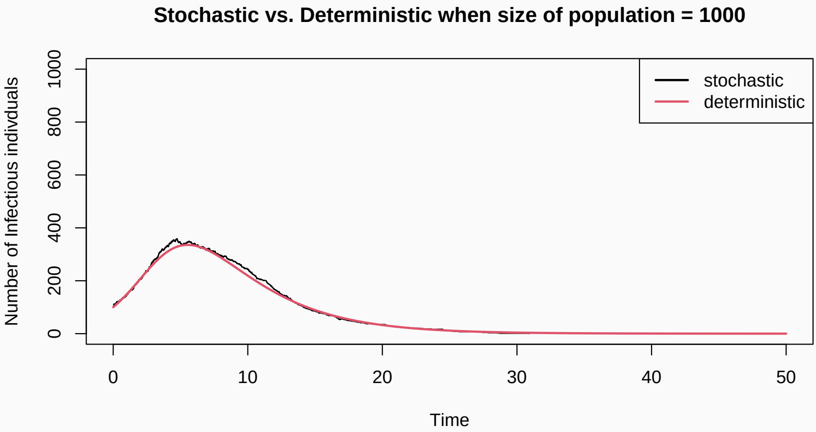
$$\beta(t) = \begin{cases} \beta_0 = 0.75 \text{ ,if } t \leq t_1 = 14, \\ \beta_1 = 0.65 * 0.75 \text{ ,if } t_1 = 14 < t \leq t_2 = 28, \\ \beta_2 = 0.75 * 0.75 \text{ ,if } t > t_2 = 28, \end{cases}$$



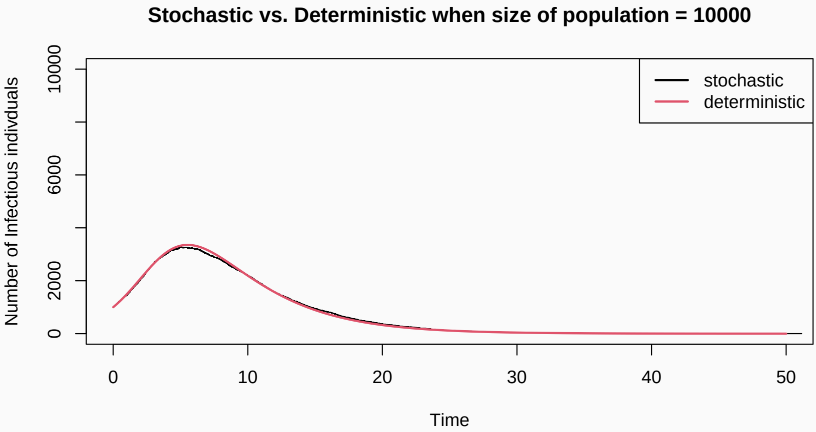
Stochastic vs. Deterministic SIR



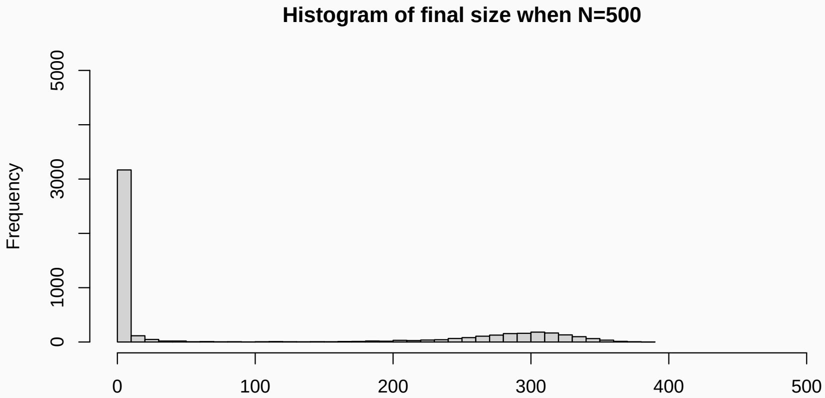
Stochastic vs. Deterministic SIR



Stochastic vs. Deterministic SIR



Histogram of Final Size



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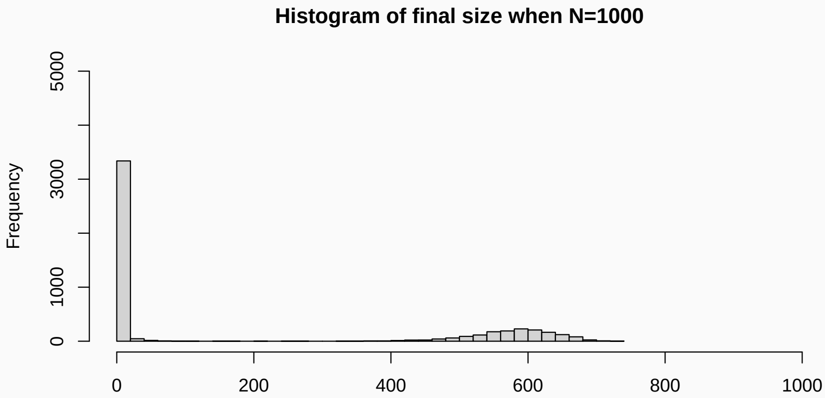
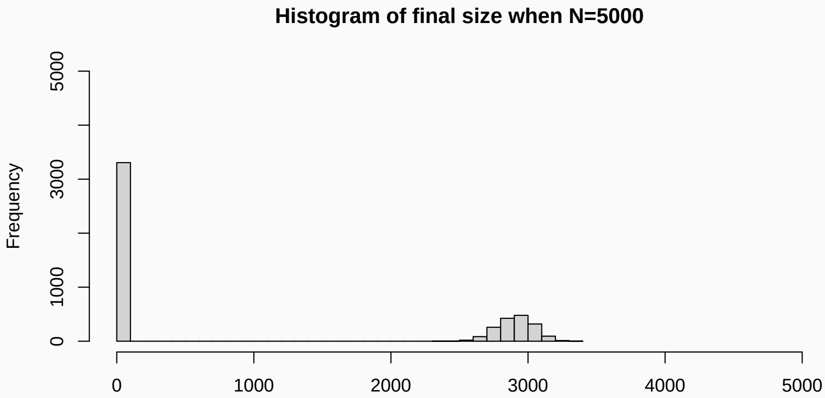


Histogram of Final Size



Exercise 1.3

Markovian SEIR Epidemic Model

- Consider a closed and homogeneous mixing population with fixed size N.
- The rate of infectious contacts is β, so the rate at which one infectious has contact with a specific other individual is β/N.
- Individuals that get infected are first latent(exposed) for a exponentially distributed period J with mean $1/\rho$, then they become infectious for a random duration $I \sim \mathrm{Exp}(\gamma)$, after which they become recovered and immune.

Markovian SEIR Epidemic Model

Let $S(t), E(t), I(t), R(t)$ be the number of individuals in four states S, E, I, R at time t respectively. Assume that $S(0) = N - 1, E(0) = 0, I(0) = 1, R(0) = 1$.

Three types of events:

- $S \to E$: a susceptible gets infected.
- $E \to I$: an exposed individual become infectious.
- $I \to R$: a infectious individual recovers.

The corresponding rates:

- $\frac{\beta}{N} S(t) I(t)$
- $\rho E(t)$
- $\gamma I(t)$

Deterministic SEIR Epidemic Model

SEIR differential equation system

$$\frac{\partial S(t)}{\partial t} = -\frac{\beta}{N} S(t) I(t),$$

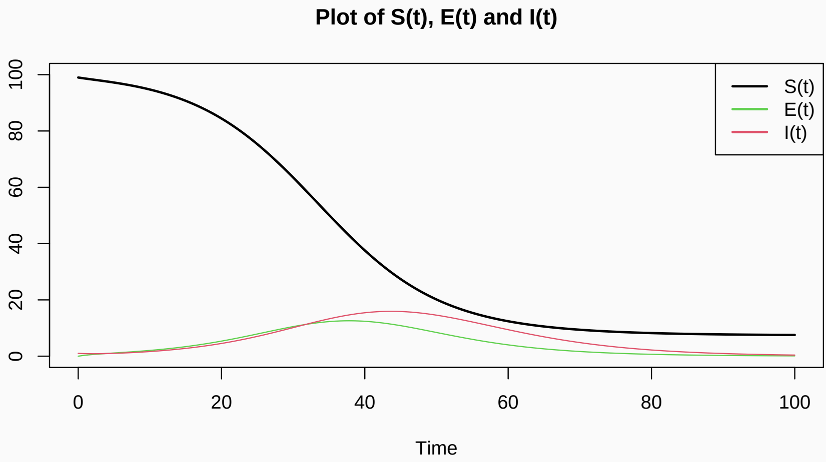
$$\frac{\partial E(t)}{\partial t} = \frac{\beta}{N} S(t) I(t) - \rho E(t),$$

$$\frac{\partial I(t)}{\partial t} = \rho E(t) - \gamma I(t).$$

Initial Conditions

$S(0) = N - 1, E(0) = 0, I(0) = 1, R(0) = 0.$

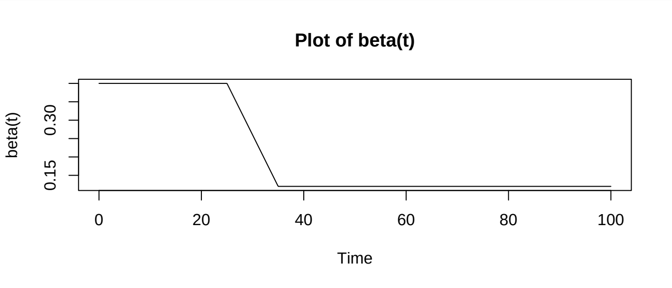
Plot the deterministic curves of $S(t)$, $E(t)$ and $I(t)$



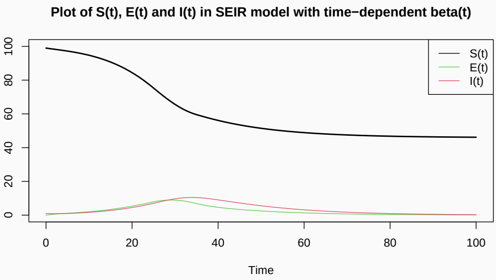
SEIR model with time-dependent $\beta(t)$

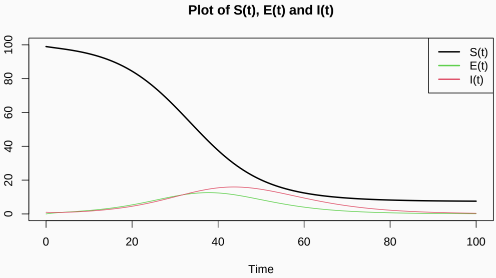
Definition of $\beta(t)$

$$\beta(t) = \begin{cases} \beta_0 \text{ ,if } t \leq t_1 - w \\ \beta_0 + \frac{\beta_1 - \beta_0}{2w} t - \frac{\beta_1 - \beta_0}{2w} (t_1 - w) \text{ ,if } t_1 - w < t \leq t_1 + w \\ \beta_1 \text{ ,if } t > t_1 + w. \end{cases}$$

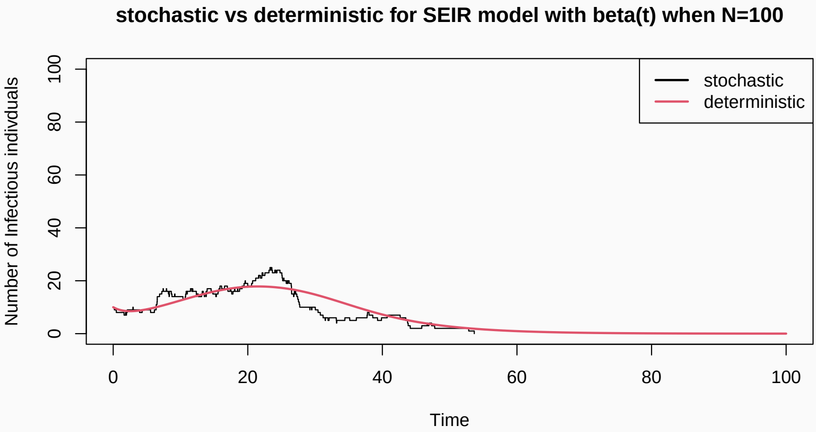


Plot of $S(t)$, $E(t)$ and $I(t)$ in SEIR model with $\beta(t)$

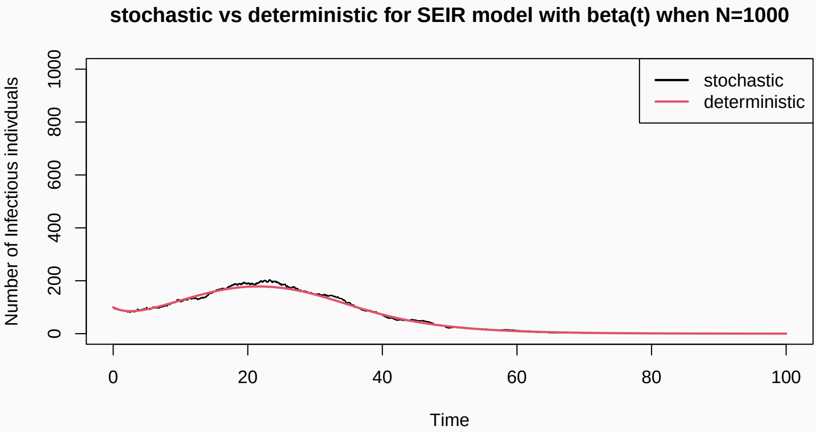




Stochastic vs. Deterministic SEIR with $\beta(t)$



Stochastic vs. Deterministic SEIR with $\beta(t)$



Stochastic vs. Deterministic SEIR with $\beta(t)$