Hypothesis Testing

Hypothesis Testing
Motivation

1. Is the chance of getting a cold different when subjects take vitamin C than when they take placebo? (Pauling 1971 data).

2. Suppose that 6 out of 15 students in a grade-school class develop influenza, whereas 20% of grade-school children nationwide develop influenza. Is there evidence of an excessive number of cases in the class?

Hypothesis Testing
Motivation

3. In a study of 25 hypertensive men we find a mean serum-cholesterol level of 220 mg/ml. In the 20-74 year-old male population the mean serum cholesterol is 211 mg/ml with standard deviation of 46 mg/ml.

- Is the mean for the population of hypertensive men also 211 mg/ml?
- Is the data consistent with that model?
- What if $\overline{X}$ = 230 mg/ml?
- What if $\overline{X}$ = 250 mg/ml?
- What if the sample was of 100 instead of 25?

Hypothesis Testing

Define:

μ = population mean serum cholesterol for male hypertensives

Hypothesis:

1. Null Hypothesis: Generally, the hypothesis that the unknown parameter equals a fixed value.

   H_0: $\mu = 211$ mg/ml

2. Alternative Hypothesis: contradicts the null hypothesis.

   H_A: $\mu \neq 211$ mg/ml

Hypothesis Testing

Decision / Action:

We assume that either H_0 or H_A is true. Based on the data we will choose one of these hypotheses.

	H_0 Correct	H_A Correct
Decide H_0	$1-\alpha$	β
Decide H_A	α	$1-\beta$

α = **"size"**
$1 - \beta$ = **"power"**

Hypothesis Testing

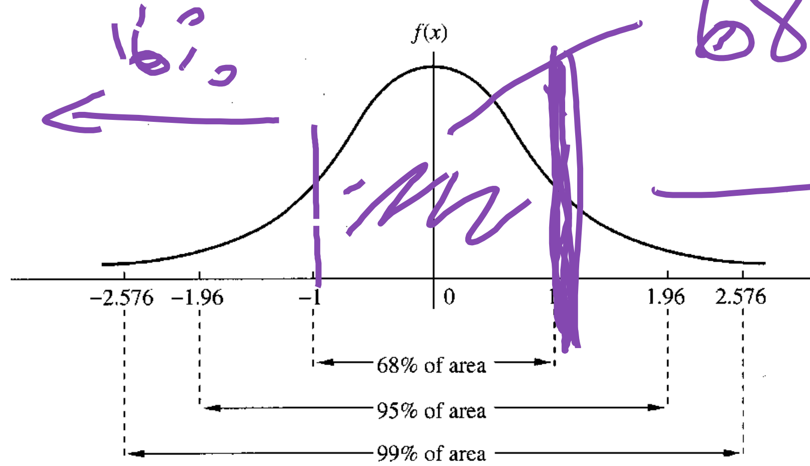
Let's fix α, for example, $\alpha = 0.05$.

$$0.05 = \alpha = P[\text{ choose } H_A \mid H_0 \text{ true }]$$

$$\alpha = P[\text{ reject } H_0 \mid H_0 \text{ true }]$$

Q: How to construct a procedure that makes this error with only 0.05 probability?

A: Suppose we assume H_0 is true and suppose that, using that assumption, the data should give us a standard normal, Z.



If $\mu = 0$ then $|Z|$ is rarely "large". A "large" $|Z|$ would make me question whether $\mu = 0$.

Hypothesis Testing

Therefore, **we reject** H_0 **if** $|Z| > 1.96$.

$\alpha = P[\text{reject } H_0 \mid H_0 \text{ true}] = 0.05$

Then if we do find a large value of $|Z|$ we can claim that:

•**Either** H_0 is true and something unusual happened (with probability α)…

•**or**, H_0 is not true.

Given α and H_0 we can construct a test of H_0 with a specified significance level. But remember, we start by assuming that H_0 is true - we haven't proved it is true. Therefore, we usually say

- $|Z| > 1.96$ then we **reject** H_0.
- $|Z| < 1.96$ then we **fail to reject** H_0.

Cholesterol Example:

Let μ be the mean serum cholesterol level for male hypertensives. We observe

$$\overline{X} = 220 \text{ mg/ml}$$

Also, we are told that for the general population...

μ_0 = mean serum cholesterol level for males = 211 mg/ml

σ = std. dev. of serum cholesterol for males = 46 mg/ml

NULL HYPOTHESIS: mean for male hypertensives is the same as the general male population.

ALTERNATIVE HYPOTHESIS: mean for male hypertensives is different than the mean for the general male population.

$$H_0 : \mu = \mu_0 = 211 \text{ mg/ml}$$

$$H_A : \mu \neq \mu_0 \;\; (\mu \neq 211 \text{ mg/ml})$$

Hypothesis Testing

Cholesterol Example:

Test H_0 with significance level α.

Under H_0 we know:

$$\frac{\overline{X} - \mu_o}{\sigma / \sqrt{n}} \sim N(0,1)$$

Therefore,

•**Reject** H_0 if $\left|\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}\right| > 1.96$ gives an $\alpha = 0.05$ test.

•**Reject** H_0 if

$$\overline{X} > \mu_0 + 1.96 \frac{\sigma}{\sqrt{n}} \quad \text{or}$$

$$\overline{X} < \mu_0 - 1.96 \frac{\sigma}{\sqrt{n}}$$

Hypothesis Testing

Cholesterol Example:

TEST: **Reject** H_0 if

$$\overline{X} > 211 + 1.96\frac{46}{\sqrt{25}} \text{ or}$$

$$\overline{X} < 211 - 1.96\frac{46}{\sqrt{25}}$$

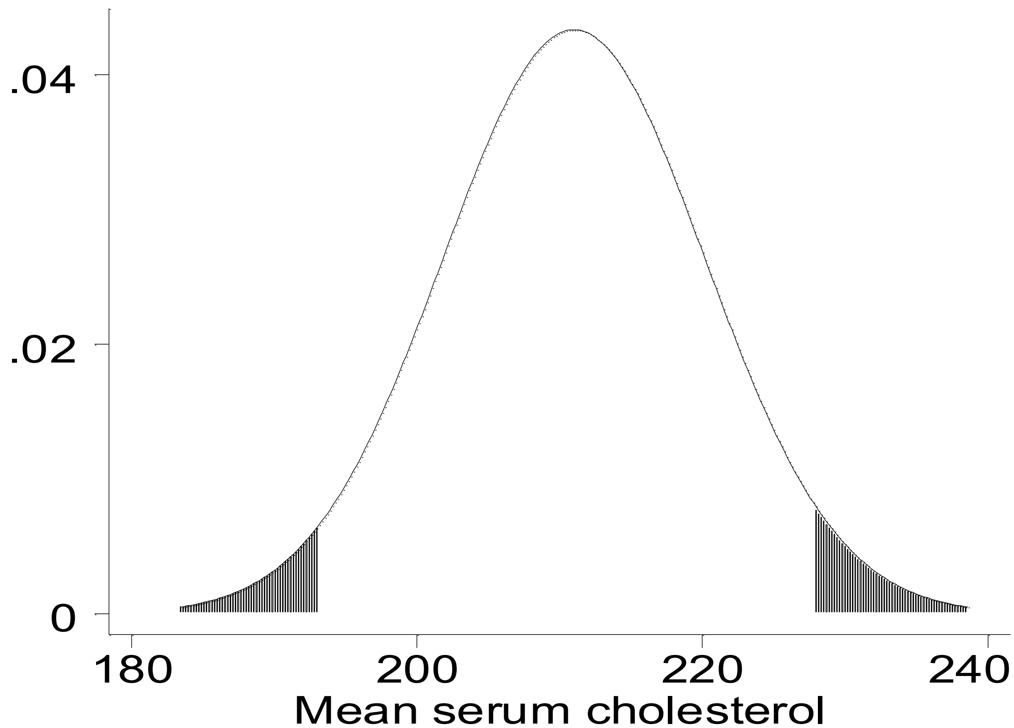
$$\overline{X} > 228.03 \text{ or}$$

$$\overline{X} < 192.97$$

In terms of Z ...

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

Reject H_0 if $Z<-1.96$ or $Z> 1.96$


.04
.02
0
180
200
220
240
Mean serum cholesterol

Hypothesis Testing

p-value:

- smallest possible α for which the observed sample would still reject H_0.
- probability of obtaining a result as extreme or more extreme than the actual sample (give H_0 true).

Hypothesis Testing

p-value: Cholesterol Example

$\overline{X} = 220$ mg/ml n = 25 $\sigma = 46$ mg/ml

$$H_0 : \mu = 211 \text{ mg/ml}$$

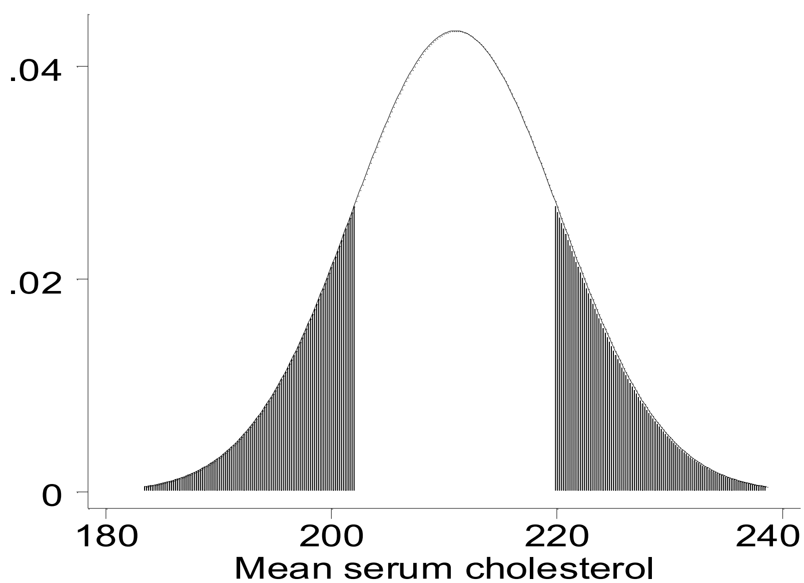
$$H_A : \mu \neq 211 \text{ mg/ml}$$

If p-value $< \alpha \implies$ reject null hypothesis

If p-value $> \alpha \implies$ do not reject null hypothesis

p-value is given by:

$$2 * P[\overline{X} > 220] = .33$$



Determination of Statistical Significance for Results from Hypothesis Tests

Either of the following methods can be used to establish whether results from hypothesis tests are statistically significant:

(1) The test statistic Z can be computed and compared with the critical value $Q_Z^{(1-\alpha/2)}$ at an α level of .05. Specifically, if H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$ are being tested and $|Z| > 1.96$, then H_0 is rejected and the results are declared *statistically significant* (i.e., $p < .05$). Otherwise, H_0 is accepted and the results are declared *not statistically significant* (i.e., $p \geq .05$). We refer to this approach as the **critical-value method**.

(2) The exact p-value can be computed, and if $p < .05$, then H_0 is rejected and the results are declared *statistically significant* . Otherwise, if $p \geq .05$ then H_0 is accepted and the results are declared *not statistically significant* . We will refer to this approach as the **p-value method** .

Guidelines for Judging the Significance of p-value

If $.05 \leq p < .10$, than the results are *marginally significant*.

If $.01 \leq p < .05$, then the results are *significant*.

If $.001 \leq p < .01$, then the results are *highly significant*.

If $p < .001$, then the results are *very highly significant*.

If $p > .1$, then the results are considered *not statistically significant* (sometimes denoted by NS).

statistical

^ Significance is not everything!

Hypothesis Testing and Confidence Intervals

Hypothesis Test: <u>Fail to reject</u> H_0 if

$$\overline{X} < \mu_0 + Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

and $$\overline{X} > \mu_0 - Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Confidence Interval: Plausible values for μ are given by

$$\mu < \overline{X} + Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

and $$\mu > \overline{X} - Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Hypothesis Testing
"how many sides?"

Depending on the alternative hypothesis a test may have a **one-sided alternative** or a **two-sided alternative**. Consider

$$H_0 : \mu = \mu_0$$

We can envision (at least) three possible alternatives

$$H_A : \mu \neq \mu_0 \qquad (1)$$

$$H_A : \mu < \mu_0 \qquad (2)$$

$$H_A : \mu > \mu_0 \qquad (3)$$

(1) is an example of a "two-sided alternative"

(2) and (3) are examples of "one-sided alternatives"

The distinction impacts

- Rejection regions
- p-value calculation

Hypothesis Testing
"how many sides?"

Cholesterol Example: Instead of the two-sided alternative considered earlier we may have only been interested in the alternative that hypertensives had a higher serum cholesterol.

$$H_0 : \mu = 211$$

$$H_A : \mu > 211$$

Given this, an $\alpha = 0.05$ test would reject when

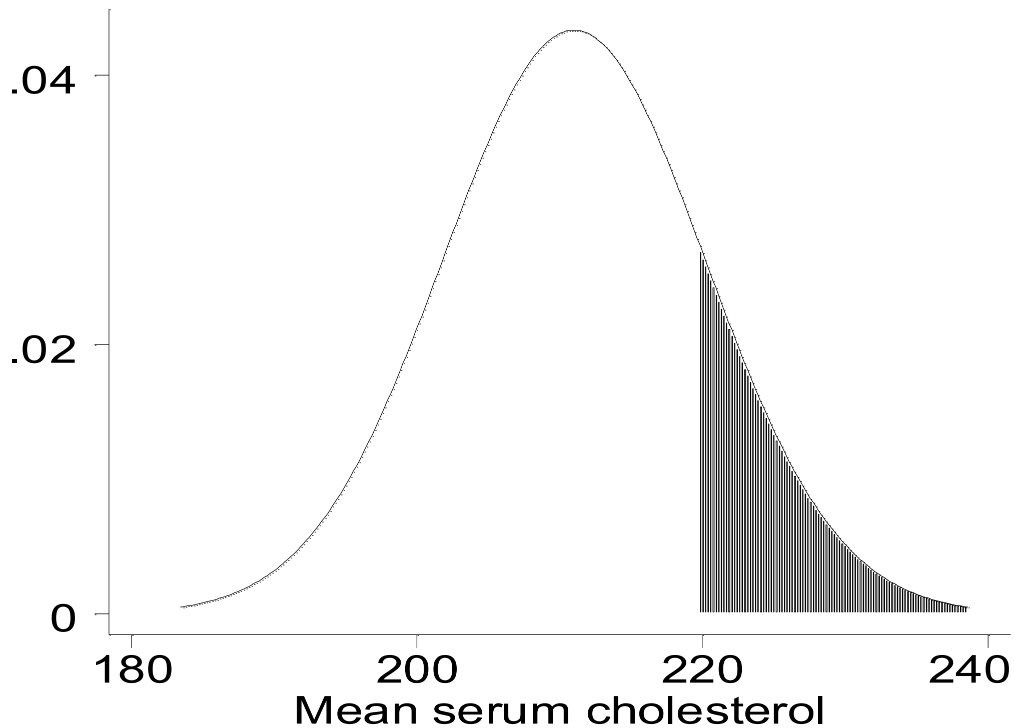
$$\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = Z > Q_Z^{(1-0.05)} = 1.65$$

We put all the probability on "one-side".

The <u>p-value</u> would be half of the previous,

$$\text{p-value} = P[\, \overline{X} > 220]$$

$$= .163$$



.04
.02
0
180
200
220
240
Mean serum cholesterol

Hypothesis Testing

Through this worked example we have seen the basic components to the statistical test of a scientific hypothesis.

Summary

1. Identify H_0 and H_A
2. Identify a test statistic
3. Determine a significance level, $\alpha = 0.05$, $\alpha = 0.01$
4. Critical value determines rejection / acceptance region
5. p-value
6. Interpret the result

American Statistical Association's guidelines on the use of p-values

1. P-values can indicate how incompatible the data are with a specified statistical model.
2. P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.
3. Scientific conclusions and business or policy decisions should not be based only on whether a p-value passes a specific threshold.
4. Proper inference requires full reporting and transparency.
5. A p-value, or statistical significance, does not measure the size of an effect or the importance of a result.
6. By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis.