Contingency Tables

Session 7

Module 1 Probability & Statistical Inference

The Summer Institutes

DEPARTMENT OF BIOSTATISTICS
SCHOOL OF PUBLIC HEALTH
UNIVERSITY of WASHINGTON

Overview

1. **Defining Categorical Variables**
    - Contingency (two-way) tables
    - χ^2 Tests
2. **Comparing Two Categorical Variables**
3. **2 x 2 Tables**
    - Sampling designs
    - Testing for association
    - Estimation of effects

DEFINITION

Factor

A **factor** is a type of variable that can take one of a small number of possible values. The possible values are called the **levels** of the factor.

Also known as a categorical variable or discrete variable.

Examples

> **Gender** with three levels:
> 1 = Male, 2 = Female, 3 = Non-binary

> **Disease status** with three levels:
> 1 = Progression, 2 = Stable, 3 = Improved

> **Age** with four levels:
> 1 = 20-29 yrs, 2 = 30-39 yrs, 3 = 40-49 yrs, 4 = 50-59 yrs

Factors and Contingency Tables

- **One-way tables** summarize the proportion of observations within each level of one factor.

- **Contingency tables**, aka **two-way tables** summarize the proportion of observations within each combination of levels from two factors.
  - Also called an **R x C** table
  - Often used to assess whether two factors are related
  - Can test whether the factors are related using a χ^2 test
  - Examining two-way tables of Factor A vs Factor B at each level of a third Factor C shows how the A/B association may be explained or modified by C (Session 8).

EXAMPLE

Categorical Data: R x C table

Doll and Hill (1952)

Retrospective assessment of smoking frequency
The table displays the daily average number of cigarettes for lung cancer patients and control patients.

⚠️ Note the equal numbers of cases and controls.

	None	< 5 cigarettes	5-14 cigarettes	15-24 cigarettes	25-49 cigarettes	50+ cigarettes	
Cases (Cancer)	7 0.5%	55 4.1%	489 36.0%	475 35.0%	293 21.6%	38 2.8%	1357
Controls (No Cancer)	61 4.5%	129 9.5%	570 42.0%	431 31.8%	154 11.3%	12 0.9%	1357
	68	184	1059	906	447	50	**2714**

EXAMPLE

Categorical Data: χ^2 test

Doll and Hill (1952)

Scientific Question

Is the distribution of smoking frequencies for those with cancer different from the distribution for those without cancer?

Restate scientific question as statistical hypotheses:

H_0: distribution of smoking same in both groups

H_A: distribution of smoking not the same

What does H_0 predict we would observe if all we knew were the marginal totals?

	None	< 5 cigarettes	5-14 cigarettes	15-24 cigarettes	25-49 cigarettes	50+ cigarettes	
Cases (Cancer)							1357
Controls (No Cancer)							1357
	68	184	1059	906	447	50	**2714**

EXAMPLE

Categorical Data: χ^2 test

Doll and Hill (1952)

Scientific Question
Is the distribution of smoking frequencies for those with cancer different from the distribution for those without cancer?

- Each group has the same proportion in each cell as the overall **marginal proportion.** The “equal” expected number for each group is the result of the equal sample size in each group.
- We can test H_0 by summarizing the difference between the observed and expected cell counts

	None	< 5 cigarettes	5-14 cigarettes	15-24 cigarettes	25-49 cigarettes	50+ cigarettes	
Cases (Cancer)	34	92	529.5	453	223.5	25	1357
Controls (No Cancer)	34	92	529.5	453	223.5	25	1357
	68	184	1059	906	447	50	**2714**

Break #1

**Pause the video,
take a break, stretch,
then review relevant exercises
from worksheet.**

Afterwards, continue on!

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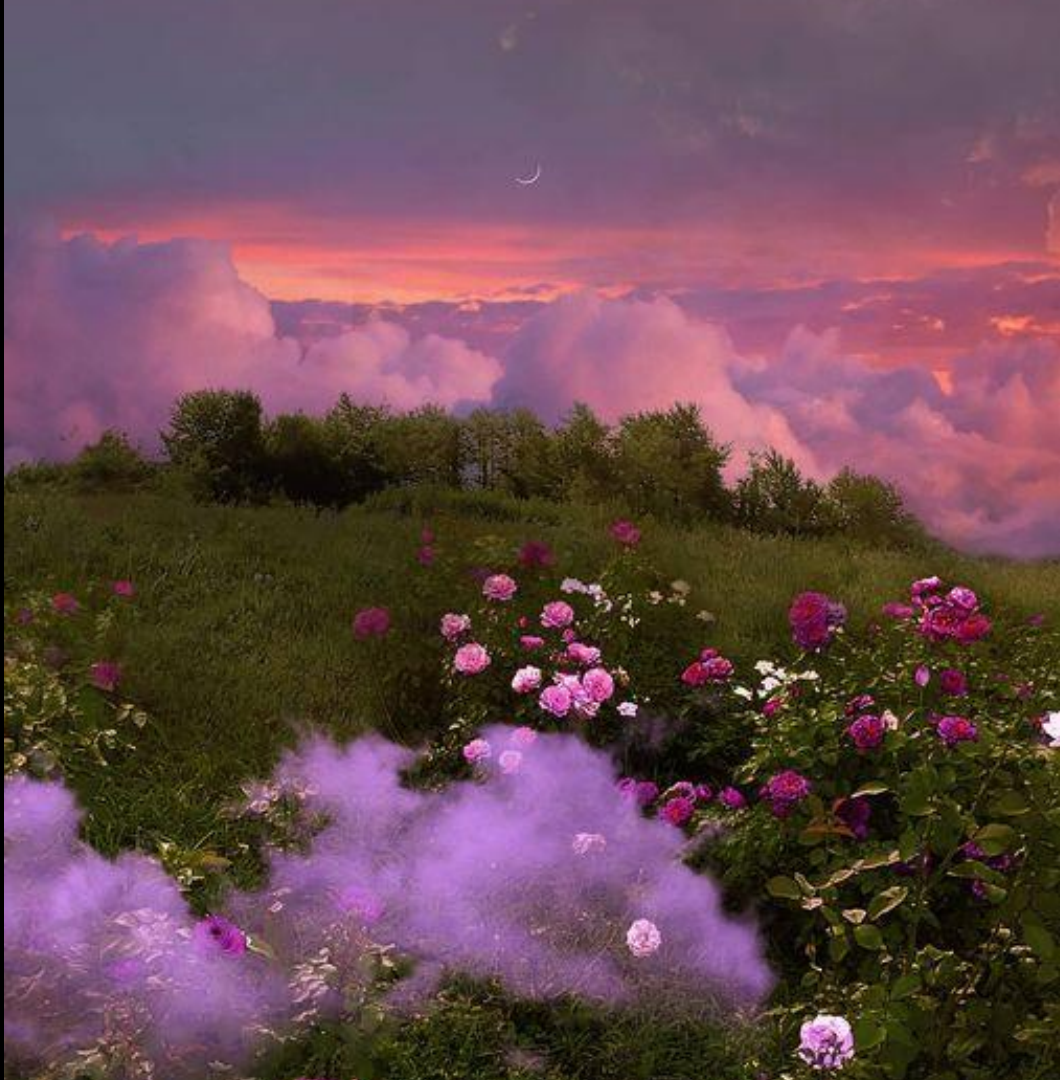

DEFINITION

Categorical Data

χ^2 Test Statistic

- ~~sum up (observed count - expected count) for all cells~~
- ~~sum up |observed count - expected count| for all cells~~

Summing the differences between the observed and expected counts provides an overall assessment of H_0.

$$X^2 = \sum_{i=1}^{R} \sum_{j=1}^{C} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2((R-1)(C-1))$$

X^2 is known as the **Pearson's Chi-square Statistic**

- Large values of X^2 suggests the data are not consistent with H_0
- Small values of X^2 suggests the data are consistent with H_0
- The χ^2 distribution approximates the distribution of X^2 when H_0 true
  - Computer intensive "exact" tests also possible

EXAMPLE

Categorical Data: χ^2 test

Doll and Hill (1952)

The contributions to the X^2 statistic are...

	None	< 5 cigarettes	5-14 cigarettes	15-24 cigarettes	25-49 cigarettes	50+ cigarettes
Cases (Cancer)	$\frac{(7-34)^2}{34} = 21.4$	$\frac{(55-92)^2}{92} = 14.9$	3.1	1.1	21.6	6.8
Controls (No Cancer)	$\frac{(61-34)^2}{34} = 21.4$	14.9	3.1	1.1	21.6	6.8

$$X^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} = 137.8 \longrightarrow$$

$\chi^2((6-1)(2-1)) =$
$\chi^2(5)$

```
pchisq(137.8, df = 5, lower.tail=FALSE)
```

p-value = $P(X^2 > 137.8 \mid H_0) < 0.0001$

🧐 **Conclusion** Reject H_0 at $\alpha = 0.05$

Categorical Data: χ^2 Test

Summary Conducting χ^2 a test

1. Compute the expected cell counts under null hypothesis (no association):

   $E_{ij} = N_i M_j / T$

2. Compute the chi-square statistic:

$$X^2 = \sum_{i=1}^{R} \sum_{j=1}^{C} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

3. Compare X^2 to $\chi^2(df)$ where

   $df = (R-1) \times (C-1)$

4. Interpret p-value

Factor Levels

Groups

	One	Two	Three	...	C	TOTAL
One	O_{11}	O_{12}	O_{13}	...	O_{1C}	N_1
Two	O_{21}					N_2
Three	O_{31}					N_3
...	...					...
R	O_{R1}				O_{RC}	N_R
TOTAL	M_1	M_2	M_3	...	M_C	T

2 x 2 Tables

Epidemiological Applications

We can write the chi-square statistic for a 2 x 2 table as

$$X^2 = \frac{N(ad - bc)^2}{n_1 \cdot n_2 \cdot m_1 \cdot m_2}$$

Compare X^2 to $\chi^2(1)$.

Disease Status

Exposure Status

	D	not D	TOTAL
E	a	b	$(a+b)=n_1$
not E	c	d	$(c+d)=n_2$
TOTAL	$(a+c)=m_1$	$(b+d)=m_2$	T

EXAMPLE

2 x 2 Tables

Epidemiological Applications: Pauling (1971)

Patients are randomized to either receive Vitamin C or placebo.
Patients are followed-up to ascertain the development of a cold.

Question 1 Is treatment with Vitamin C associated with a reduced probability of getting a cold?

Question 2 If Vitamin C is associated with reducing colds, then what is the magnitude of the effect?

Disease Status

Exposure Status	Cold	no Cold	TOTAL
Vit C	17	122	139
Placebo	31	109	140
TOTAL	48	231	279

EXAMPLE

2 x 2 Tables

Epidemiological Applications: Pauling (1971)

Scientific Q1
Is treatment with Vitamin C associated with a reduced probability of getting a cold?

Restate scientific question as statistical hypotheses:

H_0 : probability of disease does not depend on treatment

H_A : probability of disease does depend on treatment

$$X^2 = \frac{279(17 \cdot 109 - 31 \cdot 122)^2}{139 \cdot 140 \cdot 48 \cdot 231} = 4.81$$

```
pchisq(4.81, df = 1, lower.tail=FALSE)
```

p-value = $P(X^2 > 4.81 \mid H_0) = 0.028$

Conclusion Reject H_0 at $\alpha = 0.05$

Disease Status

Exposure Status	Cold	no Cold	TOTAL
Vit C	17 (12%)	122 (88%)	139
Placebo	31 (22%)	109 (78%)	140
TOTAL	48	231	279

DEFINITION

2 x 2 Tables

Epidemiological Applications: Risk Ratio

Scientific Q2
If Vitamin C is associated with reducing colds, what is the magnitude of the effect?

In the Pauling (1971) example, they fixed the number of *E* and *not E*, then evaluated the disease status after a fixed period of time (same for everyone).

This is a **prospective cohort study**.

Given this design we can estimate the **risk ratio (RR)** as $RR = \dfrac{P(D|E)}{P(D|\bar{E})} = \dfrac{p_1}{p_2}$

The range of RR is [0, ∞). The range of ln(RR) is (- ∞, +∞).

Using the natural log of RR, we're able to use a Normal approximation to calculate a confidence interval!

$$\ln\left(\widehat{RR}\right) = \ln\left(\frac{\widehat{p}_1}{\widehat{p}_2}\right) = \ln\left(\frac{a/n_1}{c/n_2}\right)$$

$$\ln\left(\widehat{RR}\right) \sim N\left[\ln\left(\frac{\widehat{p}_1}{\widehat{p}_2}\right), \frac{1-p_1}{p_1 n_1} + \frac{1-p_2}{p_2 n_2}\right]$$

$\rightarrow$

95% CI : Calculate

$$\ln\left(\widehat{RR}\right) \pm 1.96\sqrt{\frac{b}{a(a+b)} + \frac{d}{c(c+d)}}$$

then exponentiate the endpoints.

Break #2

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EXAMPLE

2 x 2 Tables

Epidemiological Applications: Keller (AJPH, 1965)

Patients with (cases) and without (controls) oral cancer were surveyed regarding their smoking frequency.

(This table collapses over the smoking frequency categories.)

Question 1 Is oral cancer associated with smoking?

Question 2 If smoking is associated with oral cancer, then what is the magnitude of the risk?

Exposure Status	Disease Status: Case	Disease Status: Control	TOTAL
Smoker	484	385	869
Non-Smoker	27	109	117
TOTAL	511	475	986

EXAMPLE

Keller (AJPH, 1965)

In this example we fixed the number of **cases** and **controls** then ascertained exposure status. Such a design is known as **case-control study**.
Based on this we are able to directly estimate:

$$P(E \mid D) \text{ and } P(E \mid \overline{D})$$

However, we are interested in the **risk ratio** of disease given exposure, which is **not estimable from these data alone** - we've fixed the number of diseased and diseased free subjects.

$$P(E \mid D) \neq P(D \mid E)$$

$$\frac{P(E \mid D)}{P(E \mid \overline{D})} \neq \frac{P(D \mid E)}{P(D \mid \overline{E})}$$

odds of exposure (conditional on having the disease)

$$\frac{P(E \mid D)/(1 - P(E \mid D))}{P(E \mid \overline{D})/(1 - P(E \mid \overline{D}))} = \frac{P(D \mid E)/(1 - P(D \mid E))}{P(D \mid \overline{E})/(1 - P(D \mid \overline{E}))}$$

DEFINITION

Odds Ratio

Instead of the risk ratio we can estimate the **exposure odds ratio** which (surprisingly) is equivalent to the **disease odds ratio**:

odds of exposure (conditional on having the disease)

$$\frac{P(E \mid D)/(1 - P(E \mid D))}{P(E \mid \overline{D})/(1 - P(E \mid \overline{D}))} = \frac{P(D \mid E)/(1 - P(D \mid E))}{P(D \mid \overline{E})/(1 - P(D \mid \overline{E}))}$$

😒 **exposure odds ratio** 🙂 **disease odds ratio**

Furthermore, for rare diseases $\begin{array}{l} 1 - P(D \mid E) \approx 1 \\ 1 - P(D \mid \overline{E}) \approx 1 \end{array}$ so the disease odds ratio approximates the risk ratio:

$$\frac{P(D \mid E)/(1 - P(D \mid E))}{P(D \mid \overline{E})/(1 - P(D \mid \overline{E}))} \approx \frac{P(D \mid E)}{P(D \mid \overline{E})}$$

🙂 **disease odds ratio** 😄 **risk ratio**

For rare diseases (i.e., prevalence <5%), **the** (sample) **odds ratio estimates the** (population) **risk ratio.**

DEFINITION

Odds Ratio

Like the risk ratio, the odds ratio ranges from [0, ∞).

$$OR = \frac{p_1(1-p_1)}{p_2(1-p_2)}$$

population odds ratio

$$\widehat{OR} = \frac{a \cdot d}{b \cdot c}$$

sample odds ratio

The **log odds ratio** has (-∞, +∞) as its range and the Normal distribution approximates its sampling distribution. Confidence intervals are based upon:

$$\ln\left(\widehat{OR}\right) \sim N\left[\ln(OR), \frac{1}{n_1 p_1} + \frac{1}{n_1(1-p_1)} + \frac{1}{n_2 p_2} + \frac{1}{n_2(1-p_2)}\right]$$

...and a **95% CI** for the log odds ratio is given by:

$$\ln\left(\frac{ad}{bc}\right) \pm 1.96\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}$$

Exponentiate the endpoints to get the CI for the odds ratio on its original scale.

Break #3

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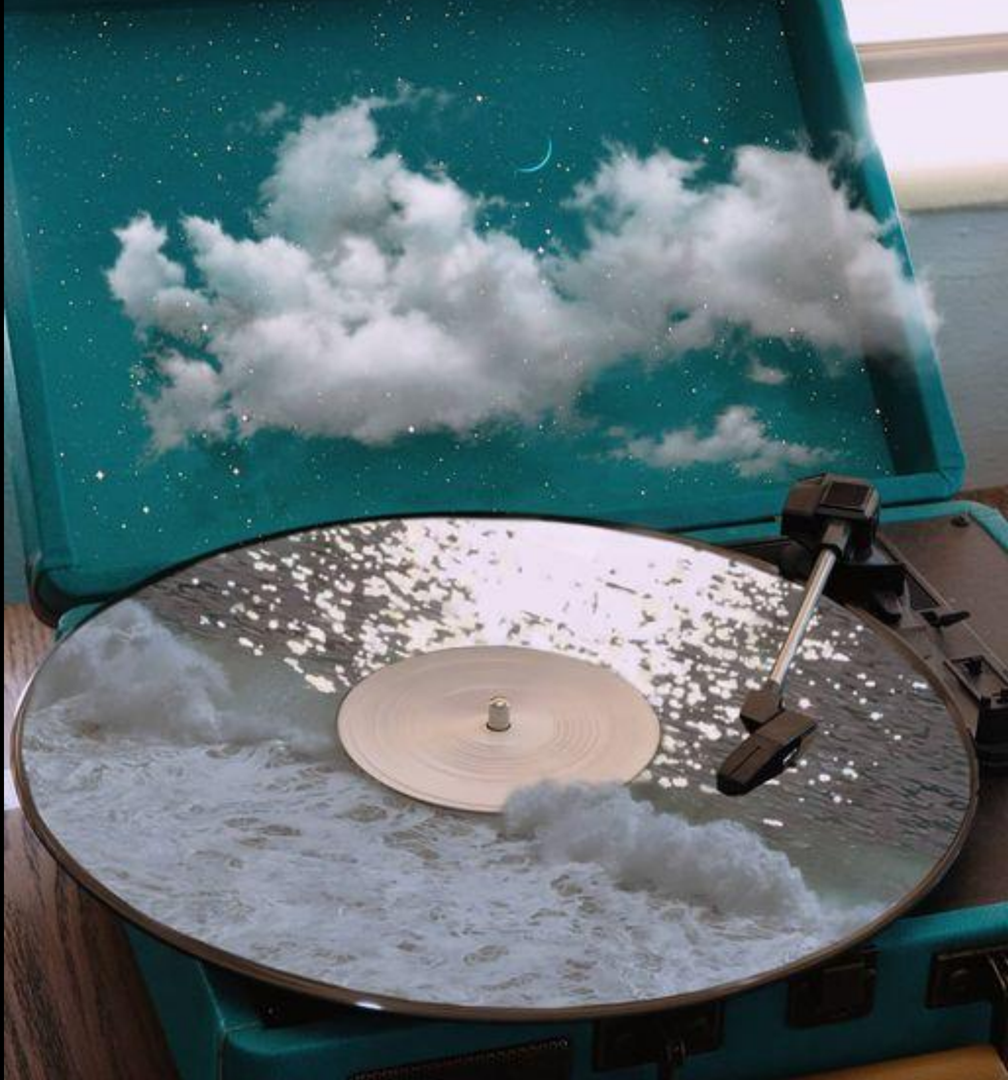

EXAMPLE

2 x 2 Tables

Epidemiological Applications: Sex-Linked Traits

Suppose we collect a random sample of Drosophila fruit flies and cross-classify by eye color and sex.

Question 1 Is eye color associated with sex?

Question 2 If eye color is associated with sex, then what is the magnitude of the effect?

Eye Color \ Sex	Male	Female	TOTAL
Red	165	300	465
White	176	81	257
TOTAL	341	381	722

EXAMPLE

2 x 2 Tables

Epidemiological Applications: Sex-Linked Traits

This is a **cross-sectional study** since only the total for the entire table is fixed in advance. The row totals or column totals are not fixed in advance.

- Sample from the entire population, not by disease status or exposure status
- Use chi-square test to test for association
- Use RR or OR to summarize association
- Cases of disease are **prevalent** cases (compared to incident cases in a prospective study.

Sex

Eye Color	Male	Female	TOTAL
Red	165	300	465
White	176	81	257
TOTAL	341	381	722

Break #4

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