Probability

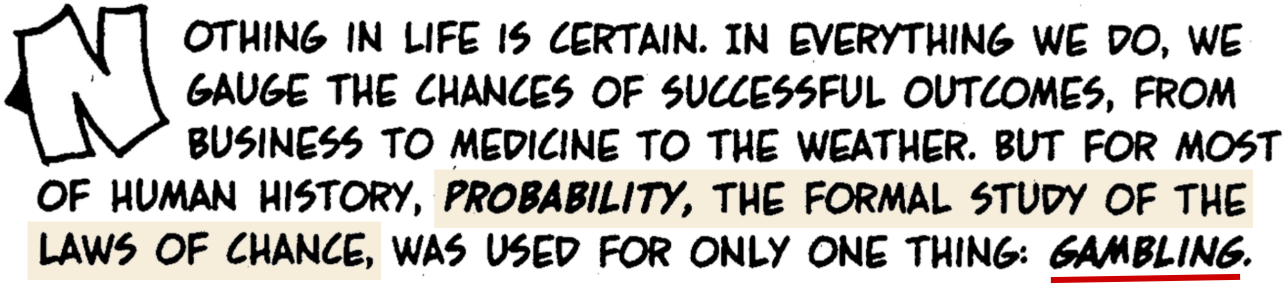
Session 1

Module 1 Probability & Statistical Inference

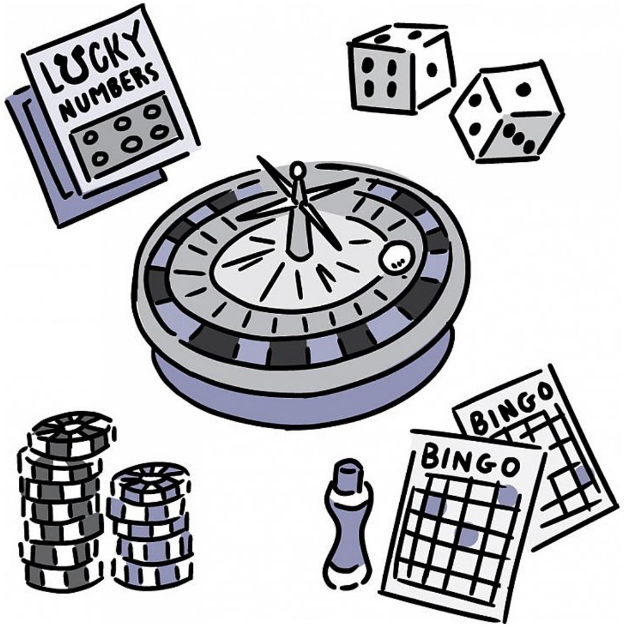
The Summer Institutes
DEPARTMENT OF BIOSTATISTICS
SCHOOL OF PUBLIC HEALTH
UNIVERSITY *of* WASHINGTON

Liber de ludo aleae *(Book on Games of Chance)*
by Gerolamo Cardano

- Written 1526 (published 1663)
- First systematic treatment of probability



> NOTHING IN LIFE IS CERTAIN. IN EVERYTHING WE DO, WE GAUGE THE CHANCES OF SUCCESSFUL OUTCOMES, FROM BUSINESS TO MEDICINE TO THE WEATHER. BUT FOR MOST OF HUMAN HISTORY, *PROBABILITY*, THE FORMAL STUDY OF THE LAWS OF CHANCE, WAS USED FOR ONLY ONE THING: *GAMBLING*.



Quotation: Larry Gonick, The Cartoon Guide to Statistics
Image credit: https://newsinhealth.nih.gov/2011/05/when-stakes-turn-toxic, https://www.sciencephoto.com/media/224088/view/coloured-portrait-of-the-italian-girolamo-cardano

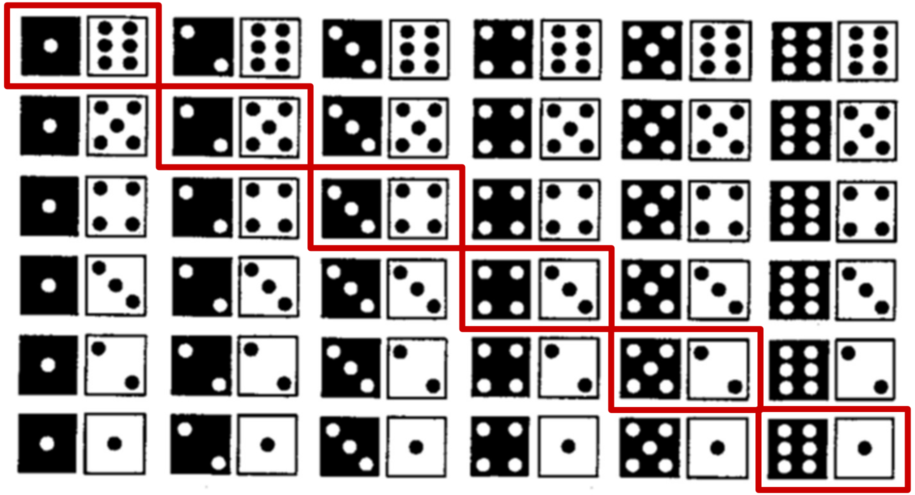
Probability

A measure of uncertainty associated with the occurrence of events or outcomes.

APPROACH 1

⭐ **Classical**: P(E) = m/N

If an event can occur in N <u>mutually exclusive,</u> <u>equally likely</u> ways, and if **m** of these possess characteristic **E**, then the probability of **E** is equal to **m/N**



Example

What is the probability of rolling a total of 7 on two dice?

E = two dice sum to 7
N = 36
m = 6

P(E) = m / N = 6/36 = 1/6

Probability

A measure of uncertainty associated with the occurrence of events or outcomes.

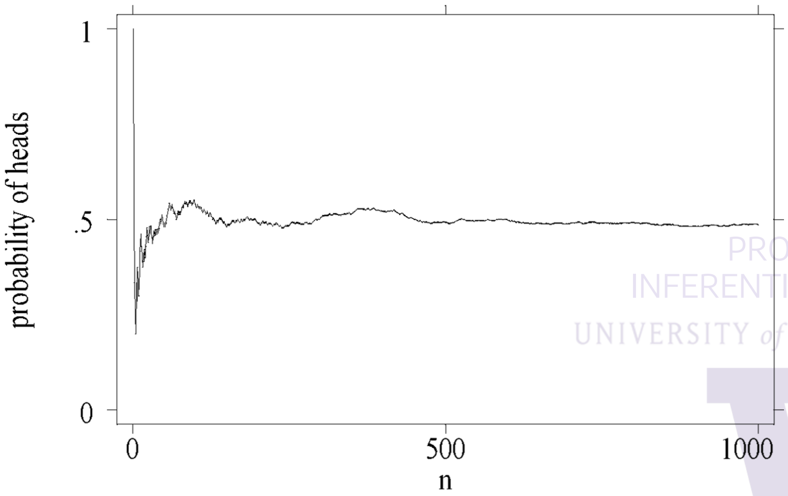
Example

Around 1900, the English statistician Karl Pearson heroically tossed a coin 24,000 times and recorded 12,012 heads, giving a proportion of 0.5005.

APPROACH 2

Relative Frequency: Pr(E) ≈ m/n

If a process or an experiment is repeated a large number of times **N**, and if the characteristic **E**, occurs m times, then the relative frequency, **m/N**, of **E** will be approximately equal to the probability of **E**.



Probability

A measure of uncertainty associated with the occurrence of events or outcomes.

APPROACH 3

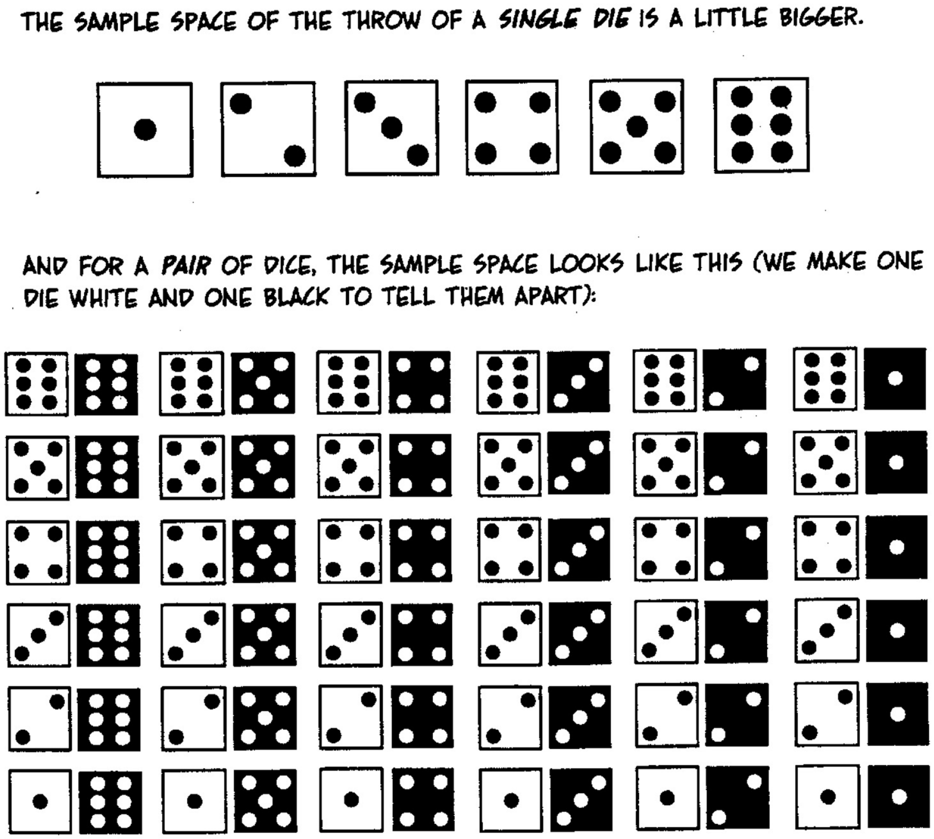
Personal Probability

What is the probability of life on Mars?

Sample Space

The **sample space** consists of the possi
An event is an outcome or set of outcor

For a coin flip the sample space is (H,T).

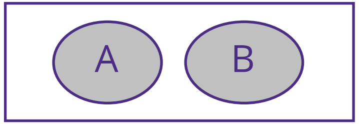


THE SAMPLE SPACE OF THE THROW OF A *SINGLE DIE* IS A LITTLE BIGGER.

AND FOR A *PAIR* OF DICE, THE SAMPLE SPACE LOOKS LIKE THIS (WE MAKE ONE DIE WHITE AND ONE BLACK TO TELL THEM APART):

Basic Properties of Probability

Two events A and B are said to be **mutually exclusive (disjoint)** if only one or the other, but not both, can occur in a particular experiment.



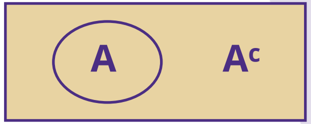
Given an experiment with n mutually exclusive events, E_1, E_2, …., E_n, the **probability of any event is non-negative and less than 1**:

$$0 \leq P(E_i) \leq 1$$

The **sum** of the probabilities of an exhaustive collection (i.e., at least one must occur) of mutually exclusive outcomes is **1**:

$$\sum_{i=1}^{n} P(E_i) = P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$

The probability of all events other than an event A is denoted by $P(A^c)$. Note that $P(A^c) = 1 - P(A)$



Notation for Joint Probabilities

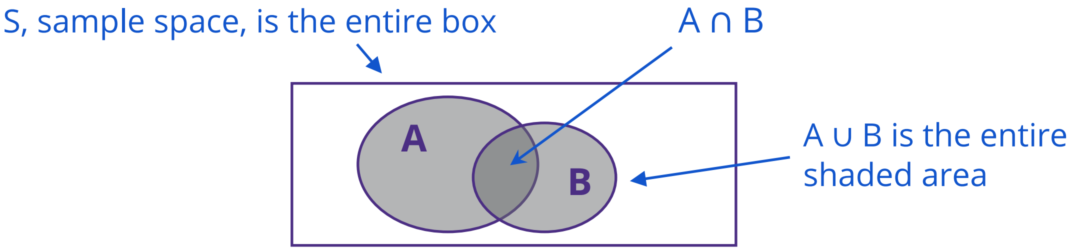
Consider that A and B are any two events.

To indicate the probability that event A **or** event B **or** both occurred, we write:

$$P(A \text{ or } B) \text{ or } P(A \cup B)$$

To indicate the probability that both A **and** B occurred, we write:

$$P(A \text{ and } B) \text{ or } P(AB) \quad \text{or } P(A \cap B)$$



S, sample space, is the entire box

A ∩ B

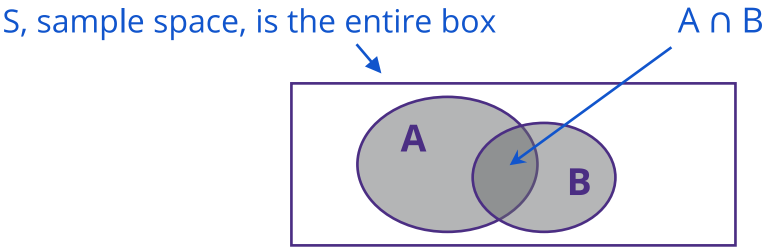
A ∪ B is the entire shaded area

Notation for Joint Probabilities

Consider that A and B are any two events.

To indicate the conditional probability for the probability of A among the subset of cases in which B is known to have occurred, we write:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



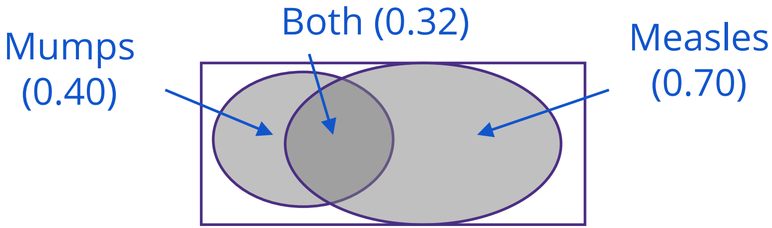
General Probability Rules

Addition Rule

Consider that A and B are any two events. The probability that event A or event B occurs (but not both) is:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

Example Of the students at Anytown High school, 40% have had the mumps, 70% have had measles and 32% have had both. What is the probability that a randomly chosen student has had at least one of the above diseases?



Note: This rule works even if A and B are mutually exclusive. If they are disjoint, then $P(AB) = 0$, and the last term drops out.

General Probability Rules

Independence

Two events A and B are said to be **independent** if and only if:

$$P(A|B) = P(A) \quad or \quad P(B|A) = P(B) \quad or \quad \boxed{P(AB) = P(A)P(B)}$$

Note: If any one holds then all three hold

Example Of the students at Anytown High school, 40% have had the mumps, 70% have had measles and 32% have had both.

Are the two events independent?

No, because P(mumps and measles) = 0.32 while P(mumps) P(measles) = 0.28

Note: This rule demonstrates that *when events are independent*, the probability of A and B occurring is given by product of their individual probabilities.

Time for a paws?

Work through questions 1-4

General Probability Rules

Multiplication Rule

Two events A and B are likewise independent if:

$$P(AB) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Under independence,
$P(A|B) = P(A)$

Under independence,
$P(B|A) = P(B)$

Example Of the students at Anytown High school, 40% have had the mumps, 70% have had measles. The probability of having measles given you have mumps is 80%.

What's the probability of having both?

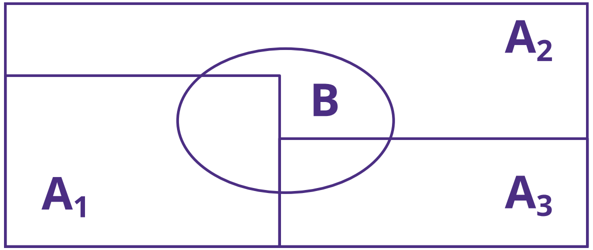
P(measles and mumps) = P(measles | mumps) P(mumps) = 0.80 * 0.40 = 0.32

General Probability Rules

Total Probability

If $A_1, \ldots, A_n$ are mutually exclusive, exhaustive events, then:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i)$$



General Probability Rules

Bayes’ Rule

Bayes’ rule combines the multiplication rule with the total probability rule:

$$P(A_j \,|\, B) = \frac{A_j \cap B}{P(B)} = \frac{P(B\,|A_j)\,P(A_j)}{P(B)} = \frac{P(B\,|A_j)\,P(A_j)}{\sum_{i=1}^{n} P(B\,|\,A_i)\,P(A_i)}$$

In the situation where A and B have two levels each, e.g., A and A^c, B and B^c, then the formula becomes:

$$P(A\,|\,B) = \frac{P(B\,|A)\,P(A)}{P(B\,|\,A)\,P(A) + P(B\,|\,A^c)\,P(A^c)}$$

EXAMPLE

An Application of Bayes’ Rule

Screening

Suppose we have a random sample of 1100 people from a population…

A = disease pos.

B = test pos.

Prevalence = $P(A) = 100/1100 = 0.091$

Sensitivity = $P(B \mid A) = 90/100 = 0.9$

Specificity = $P(B^c \mid A^c) = 970/1000 = 0.97$

PVP = $P(A \mid B) = 90/120 = 0.75$
(predictive value of a positive test)

PVN = $P(A^c \mid B^c) = 970/980 = 0.99$
(predictive value of a negative test)

		Disease Status		
		Positive	Negative	
Test Result	Positive	90	30	120
	Negative	10	970	980
		100	1000	1100