# Pathway & Network Analysis of Omics Data: Introduction

Ali Shojaie
Department of Biostatistics
University of Washington
faculty.washington.edu/ashojaie

Summer Institute for Statistical Genetics – 2023

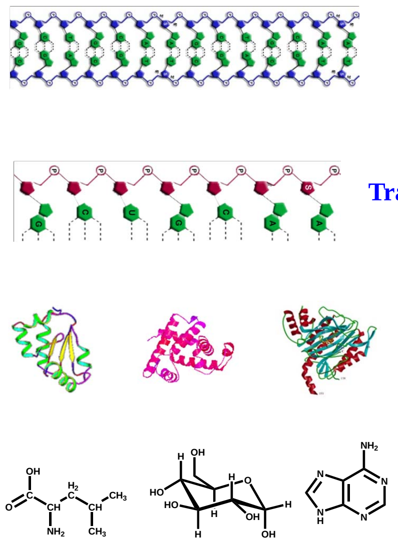




## Why Study Networks?

- Components of biological systems (genes, proteins etc) interact with each other to carry out cell functions.
- Examples of such interactions include signaling, regulation and interactions between proteins.
- We cannot understand the function and behavior of biological systems by studying individual components ($2 + 2 \neq 4!$).
- Networks provide an efficient representation of complex interactions in cells, and a basis for mathematical/statistical models to study these systems.

# Central Dogma of Molecular Biology (Extended)

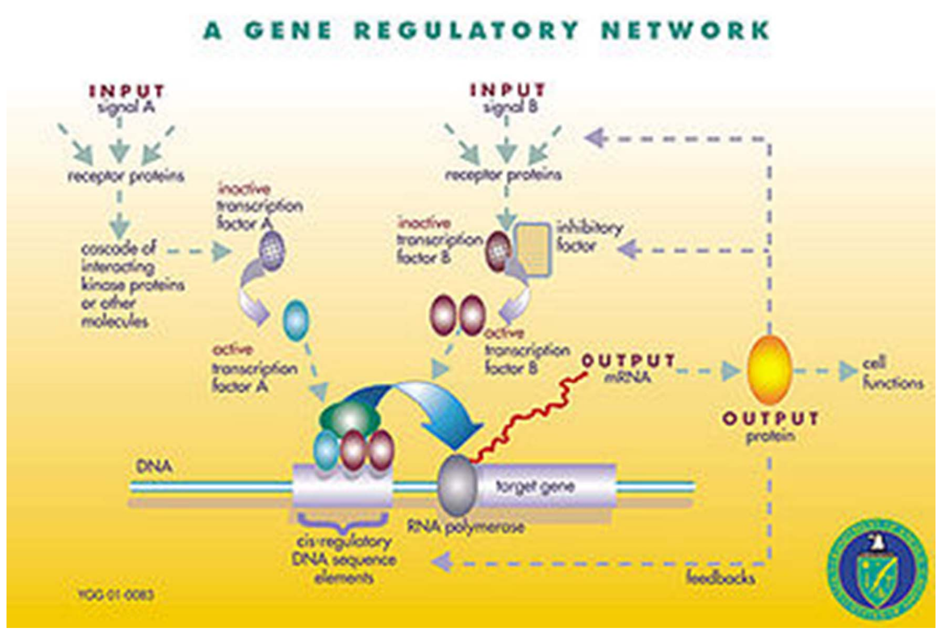


| | | |
|---|---|---|
| DNA | | Genomics – 25,000 Genes |
| ↓ | | ↓ |
| RNA | | Transcriptomics – 100,000 Transcripts |
| ↓ | | ↓ |
| Protein | | Proteomics – 1,000,000 Proteins |
| ↓ | | ↓ |
| Biochemicals (Metabolites) | | Metabolomics ~2000 Compounds |





# Networks in Biology: Gene Regulatory Interactions



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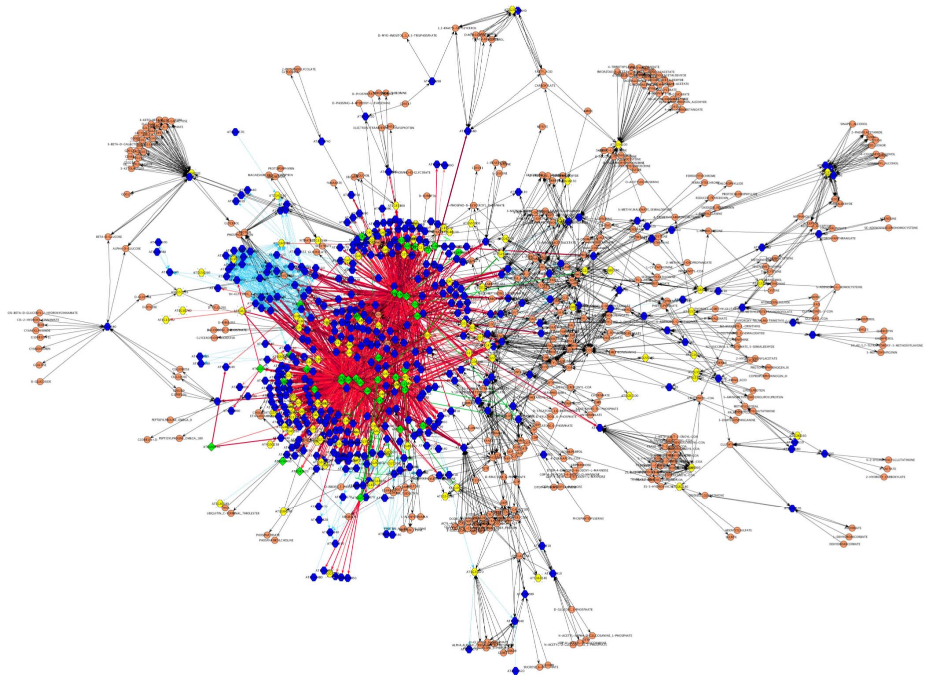
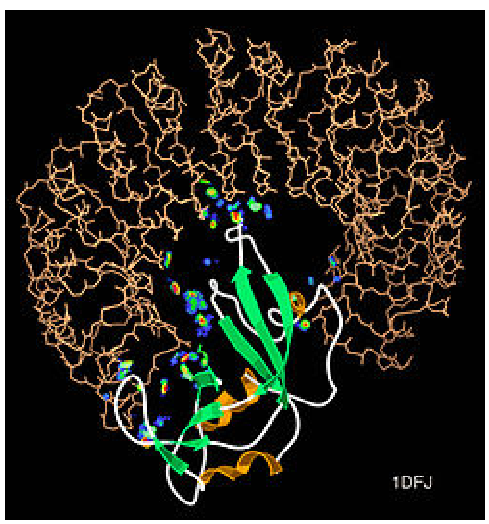


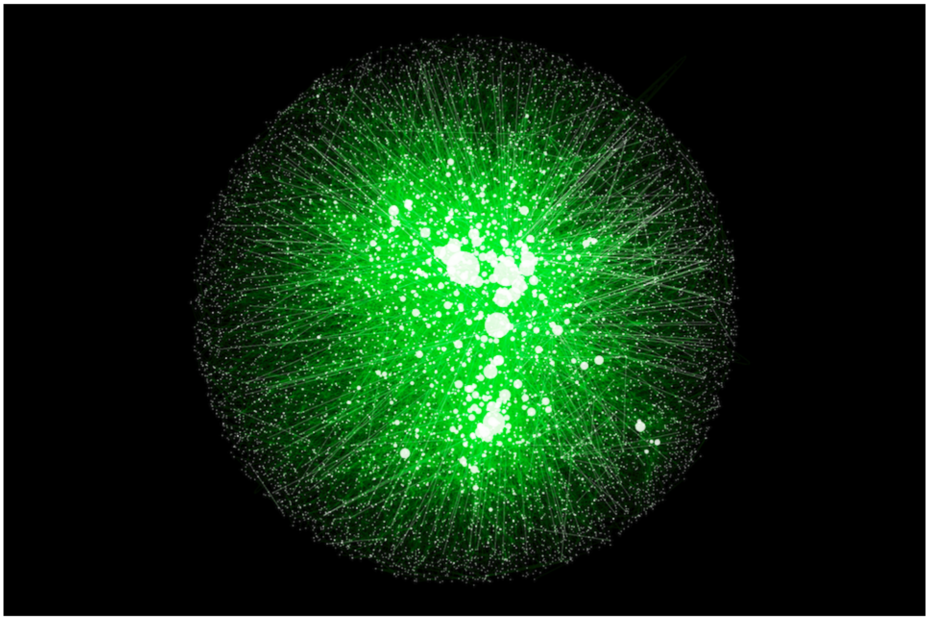




# Networks in Biology: Protein-Protein Interaction



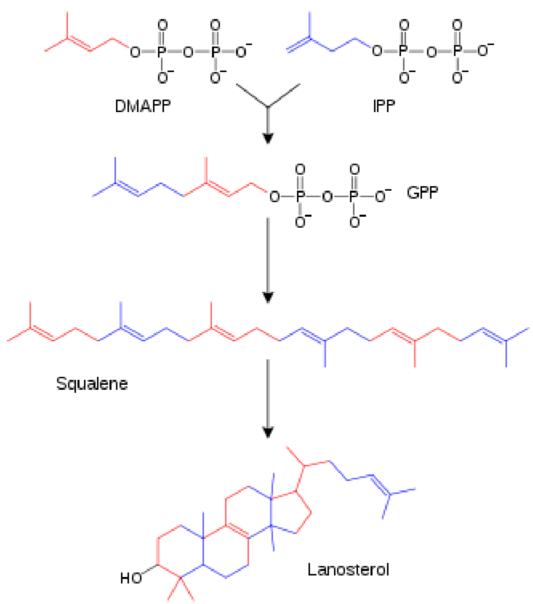
# Networks in Biology: Protein-Protein Interactions (PPI)



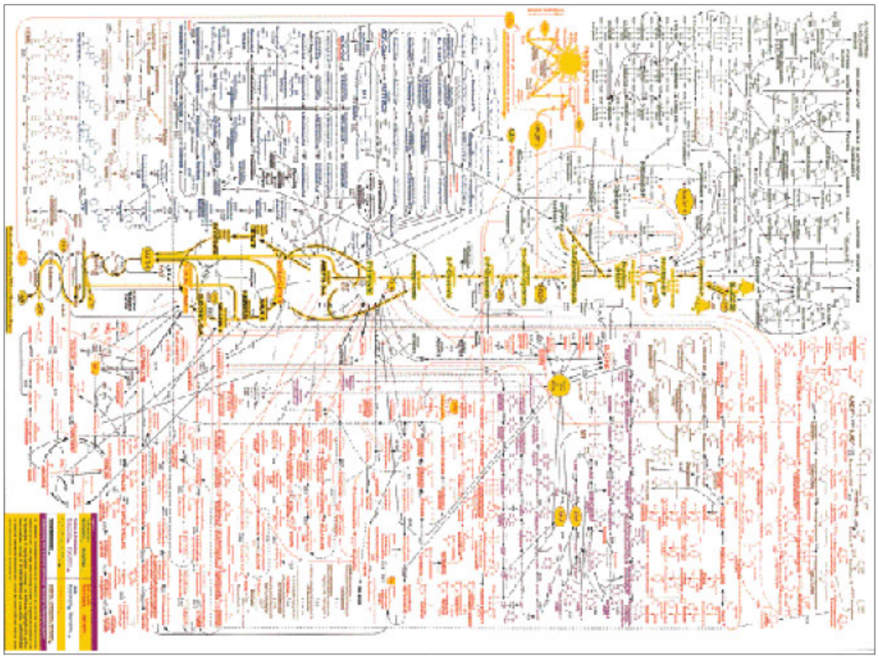


Networks in Biology
Statistical Models for Network Analysis

# Networks in Biology: Metabolic Reactions



# Networks in Biology: Metabolic Pathways







# But Do Networks Matter?

- They Do!
- Recent studies have linked changes in gene/protein networks with many human diseases.

**Systems Biology and Emerging Technologies**

## Gene Networks and microRNAs Implicated in Aggressive Prostate Cancer

**Liang Wang,[1] Hui Tang,[2] Venugopal Thayanithy,[3] Subbaya Subramanian,[3] Ann L. Oberg,[2] Julie M. Cunningham,[1] James R. Cerhan,[2] Clifford J. Steer,[4] and Stephen N. Thibodeau[1]**

[1]Departments of Laboratory Medicine and Pathology and [2]Health Sciences Research, Mayo Clinic, Rochester, Minnesota; and Departments of [3]Laboratory Medicine and Pathology, [4]Medicine, and Genetics, Cell Biology, and Development, University of Minnesota, Minneapolis, Minnesota

# But Do Networks Matter?

0888-8809/07/$15.00/0
*Printed in U.S.A.*

Molecular Endocrinology 21(9):2112–2123

doi: 10.1210/me.2006-0474

## Estrogen-Regulated Gene Networks in Human Breast Cancer Cells: Involvement of E2F1 in the Regulation of Cell Proliferation

Joshua D. Stender, Jonna Frasor, Barry Komm, Ken C. N. Chang, W. Lee Kraus, and Benita S. Katzenellenbogen

*Departments of Biochemistry (J.D.S.) and Molecular and Integrative Physiology (J.F., B.S.K.), University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3704; Women's Health and Musculoskeletal Biology (B.K., K.C.N.C.), Wyeth Research, Collegeville, Pennsylvania 19426; and Department of Molecular Biology and Genetics (W.L.K.), Cornell University, Ithaca, New York 14853-4203*





# But Do Networks Matter?

Cell PRESS

Cancer Cell
**Article**

## A Transcriptional Signature and Common Gene Networks Link Cancer with Lipid Metabolism and Diverse Human Diseases

**Heather A. Hirsch,[1,7] Dimitrios Iliopoulos,[1,7] Amita Joshi,[1,7] Yong Zhang,[2] Savina A. Jaeger,[3] Martha Bulyk,[3,4,5] Philip N. Tsichlis,[6] X. Shirley Liu,[2] and Kevin Struhl[1,*]**
[1]Department of Biological Chemistry and Molecular Pharmacology, Harvard Medical School, Boston, MA 02115, USA
[2]Department of Biostatistics and Computational Biology, Dana Farber Cancer Institute, Harvard School of Public Health, Boston, MA 02115, USA
[3]Division of Genetics, Department of Medicine, Brigham and Women's Hospital and Harvard Medical School, Boston, MA 02115, USA
[4]Department of Pathology, Brigham and Women's Hospital and Harvard Medical School, Boston, MA 02115, USA
[5]Harvard/MIT Division of Health Sciences and Technology (HST), Harvard Medical School, Boston, MA 02115, USA
[6]Molecular Oncology Research Institute, Tufts Medical Center, Boston, MA 02111, USA
[7]These authors contributed equally to this work
*Correspondence: kevin@hms.harvard.edu
DOI 10.1016/j.ccr.2010.01.022

# But Do Networks Matter?

And, incorporating the knowledge of networks improves our ability to find causes of complex diseases.

Molecular Systems Biology 3; Article number 140; doi:10.1038/msb4100180
Citation: *Molecular Systems Biology* 3:140

www.molecularsystemsbiology.com

molecular systems biology

**REPORT**

## Network-based classification of breast cancer metastasis

**Han-Yu Chuang[1,5], Eunjung Lee[2,3,5], Yu-Tsueng Liu[4], Doheon Lee[3] and Trey Ideker[1,2,4,*]**

[1] Bioinformatics Program, University of California San Diego, La Jolla, CA, USA, [2] Department of Bioengineering, University of California San Diego, La Jolla, CA, USA, [3] Department of Bio and Brain Engineering, Korea Advanced Institute of Science and Technology, Daejeon, Korea and [4] Cancer Genetics Program, Moores Cancer Center, University of California San Diego, La Jolla, CA, USA
[5] These authors contributed equally to this work
* Corresponding author. Department of Bioengineering, University of California San Diego, La Jolla, CA 92093, USA. Tel.: +1 858 822 4558; Fax: +1 858 534 5722; E-mail: trey@bioeng.ucsd.edu





# Networks: A Short Primer

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- A network is a collection of nodes $V$ and edges $E$.



Networks in Biology
Statistical Models for Network Analysis

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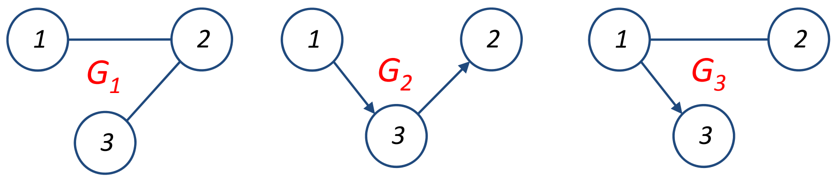




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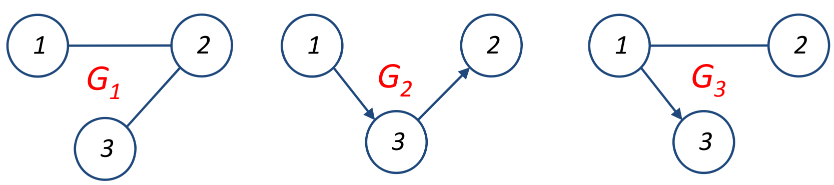


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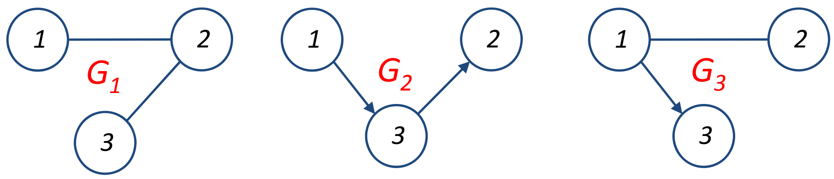


- In all these example, the node set is $V = \{1, 2, 3\}$.
- The edges are:

$$E_1 = \{1 - 2, 2 - 3\}$$
$$E_2 = \{1 \to 3, 3 \to 2\}$$
$$E_3 = \{1 - 2, 1 \to 3\}$$



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- For undirected edges, we add a **1** in both directions





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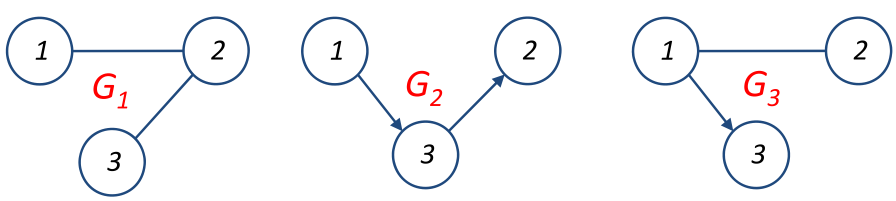

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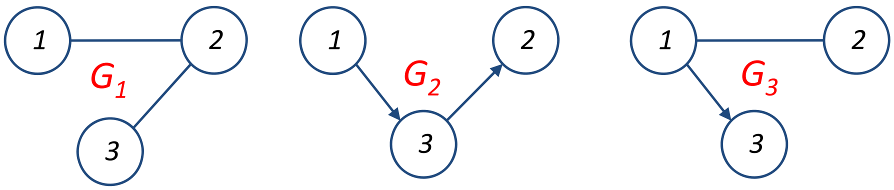


$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 \end{bmatrix} A = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 \end{bmatrix} A = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





## What Do Edges in Biological Networks Mean?

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- In metabolic networks, an edge between compound $i$ and $j$ often means that *the two compounds are involved in the same reaction*, meaning that they are generated at relative rates.
- Thus, edges represent some type of association among genes, proteins or metabolites, defined generally to include *linear or nonlinear* associations; more later....





## Statistical Models for Biological Networks

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Networks in Biology
Statistical Models for Network Analysis

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- In practice, we observe $n$ measurements of each of the variables (genes/proteins/ metabolites) for say different individuals, and want to determine which variables are connected, or use their connection for statistical analysis

# Our Plan

We will cover the following topics

- Methods for detecting signal on known networks
  - Network analysis based on centrality and clustering
  - Topology-based pathway enrichment analysis
- Methods for learning undirected networks
  - Co-expression networks
  - ARACNE
  - Conditional independence graphs
    - Gaussian observations (`glasso`, etc)
    - Non-Gaussian and non-linear data (`nonparanormal`, etc)
- Methods for learning directed networks
  - Bayesian Networks (basic concepts, reconstruction algorithm)
  - Learning directed networks from time-course data (dynamic Bayesian networks)

# Pathway & Network Analysis of Omics Data: Analysis of Network-Structured Data

Ali Shojaie
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## Introduction

Suppose we observe activities of individual nodes (genes, proteins, brain regions, etc) on a network (gene regulatory network, structural connectivity network, etc)

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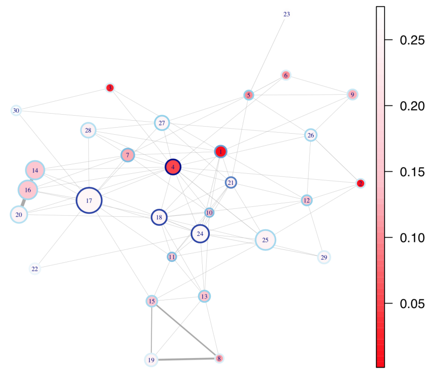






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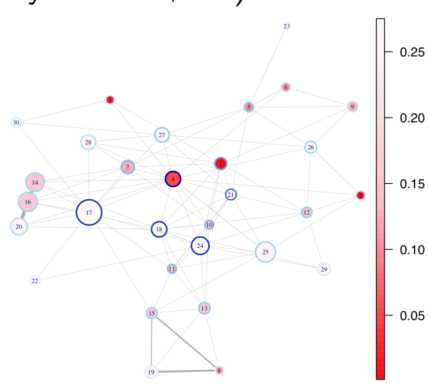


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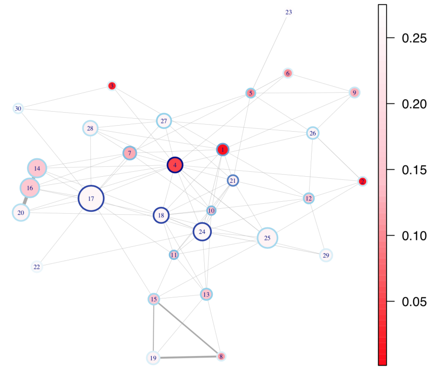


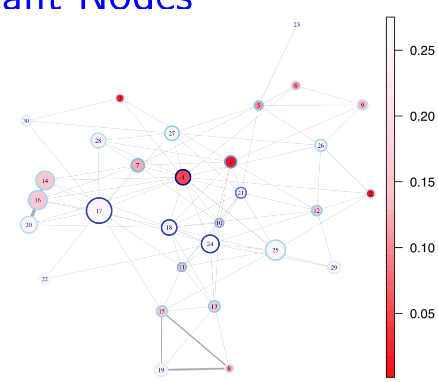

How can we identify the important nodes?

*and what does this even mean*?





# Identifying Important Nodes

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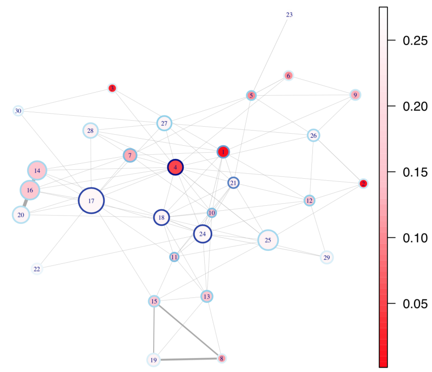


How can we identify the important nodes?

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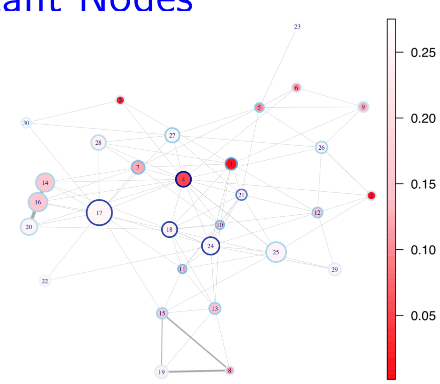


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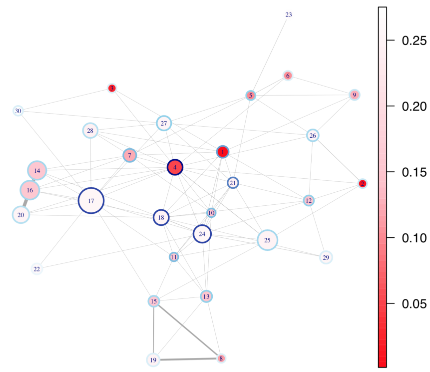


How can we identify the important nodes?

- We can select the significant nodes based on p-values, after adjusting for multiple comparisons (FDR, etc)
- But the signal is often weak for lots of tests
- If we believe the network is informative, it may make sense to use the network to guide our selection



**Introduction**
Signal Detection on Networks
Topology-Based Pathway Enrichment Analysis
De-Novo Identification of Enriched Modules

## Identifying Important Nodes

Possible strategies:

- Identify individual nodes associated with the outcome by incorporating the network (signal detection on network)
- Test if (pre-specified) subnetworks are associated with the outcome (topology-based pathway enrichment analysis)
- Identify collections of (connected) nodes that are associated with the outcome (*de-novo identification of enriched modules*)

# Signal Detection on Networks





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The simplest option is to limit our search/testing to the central nodes in the network:





## Signal Detection on Networks

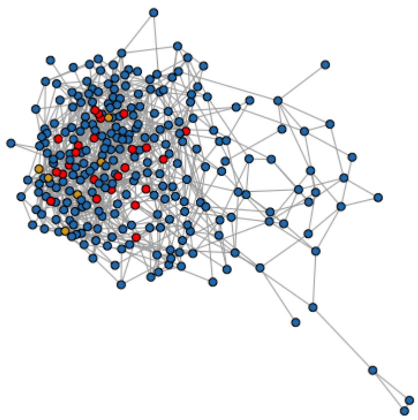
How can we identify the important nodes in a network?

The simplest option is to limit our search/testing to the central nodes in the network:

- Nodes connected to many other nodes, aka hub nodes
- Nodes that are close to many other nodes (closeness)
- Nodes that are on many network paths (betweenness)

## Example: Functional Relevance of Hub Nodes

- Inferred genetic interaction network of cancer-related pathway in prostate cancer (data from TCGA)
- Hubs defined as nodes whose degrees are at the 75th percentile of the degree distribution







## Other Measures of Centrality

- Closeness: Total distance of each node to other nodes:

$$\mathrm{cl}_j = \left( \sum_{k \in V} d(j,k) \right)^{-1}$$

where $d(j,k)$ is the (shortest path) distance between $j$ and $k$.

- Betweenness: The number of *paths* that go through a node:

$$\mathrm{bw}_j = \sum_{i \neq j \neq k} \frac{\pi_{ik}(j)}{\pi_{ik}}$$

where $\pi_{ik}(j)$ is the number of paths between $i$ and $k$ that go through $j$, and $\pi_{ik}$ is the total number of paths between them.

# Identifying "Central" Nodes

Calculating centrality measures using `igraph`:

- Hub nodes: `hub_score(graph)`
- Closeness: `closeness(graph, vids)`
  - use `estimate_closeness()` for larger networks)
- Betweenness: `betweenness(graph, vids)`
  - use `estimate_betweenness()` for larger networks

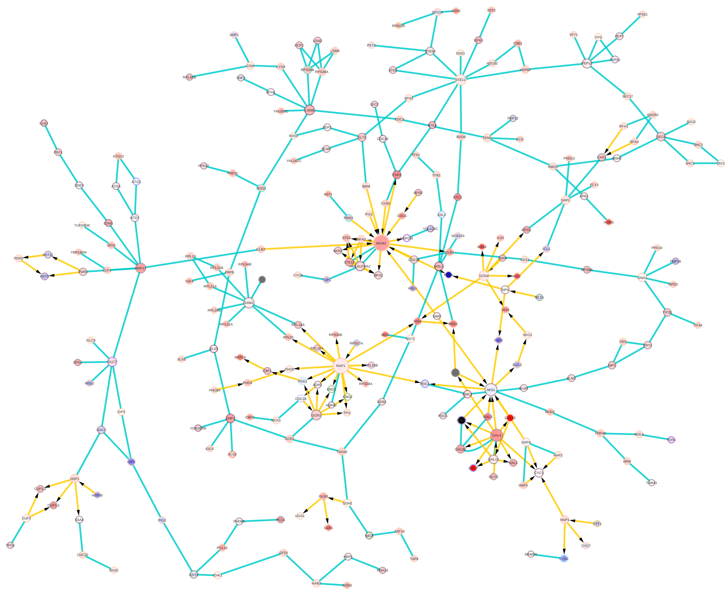




# Topology-Based Pathway Enrichment Analysis

# Yeast GAL Pathway

Ideker et al, 2001







# Topology-Based Pathway Enrichment Analysis

Test for changes in activities of node (genes, brain ROIs, etc) in pre-specified subnetworks, while incorporating network information

Two possible null hypotheses:

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Test for changes in activities of node (genes, brain ROIs, etc) in pre-specified subnetworks, while incorporating network information

Two possible null hypotheses:

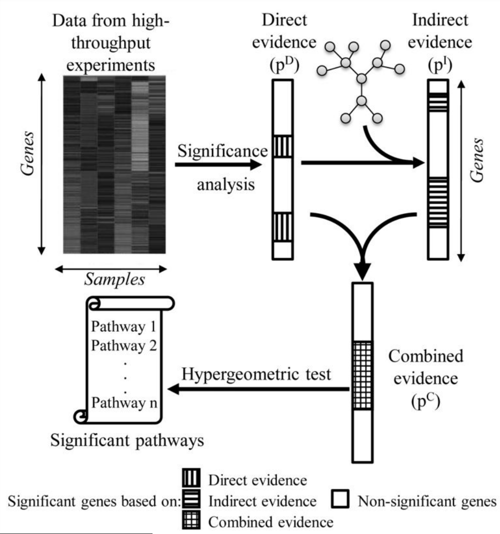
- Competitive null hypothesis: activity of each pathway is compared with other pathways, often using a permutation test
  - Assume few genes are differentially connected, and may be sensitive to the choice of gene sets
- Self-contained null hypothesis: activity of each pathway is compared against the null distribution
  - More rigorous, but may be sensitive to modeling assumptions (*Goemen & Buhlmann* (07), *Ackermann & Strimmer* (09))





# PathNet[1]

A simple topology-based pathway enrichment method:




[1] Dutta et al (2012)

## PathNet: Details

- Each gene's $p$-value from differential expression is combined with $p$-values of its neighbors using Fisher's methods

$$\mathrm{SI}_j = \sum_{k \in \mathsf{ne}(j)} \left\{ -\log_{10}\left(p_k^D\right) \right\}.$$

  - The indirect $p$-value, $p^I$ is calculated from $\mathrm{SI}_j$ by permutation
- Direct ($p_j^D$) and indirect ($p_j^I$) p-values are then combined ($p_j^C$)
- The significance of $p_j^C$ for genes in each pathway is assessed using a hypergeometric test
- Implemented in Bioconductor package `PathNet`





## topologyGSA[2]

- topologyGSA (Gene Set Analysis Exploiting Pathway Topology) assumes that data are normally distributed:

$$X^1 \sim N(\mu^1, \Sigma^1), \quad X^2 \sim N(\mu^2, \Sigma^2)$$

- It obtains estimates of $\Sigma^1$ and $\Sigma^2$ based on the networks (think graphical lasso, but with known nonzero entries)
- It then performs two tests:
  - equality of covariance matrices: $H_0^c : \Sigma^1 = \Sigma^2$
  - equality of means $H_0^m : \mu^1 = \mu^2$ — it uses different methods depending on the result of $H_0^c$
- Implemented in R-package `topologyGSA` (also in `graphite`)

[2] Massa et al (2010)

## Signaling Pathway Impact Analysis (SPIA)[3]

- Combines overrepresentation analysis (ORA) with measure of perturbation of a given pathway under a given condition
- A bootstrap procedure is used to assess the significance of the observed pathway perturbation (difficult to extend to comparison of $> 2$ conditions)
- Currently not applicable to all pathways (more later)
- Analyzes each pathway separately (ignores connections between pathways)
- Implemented in the Bioconductor package `SPIA`

[3] Tarca et al (2009)





## The SPIA Methodology

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SPIA combines two types of evidence



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(i) the overrepresentation of DE genes in a given pathway

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SPIA combines two types of evidence

(i) the overrepresentation of DE genes in a given pathway

- measured by the p-value for the given number of DE genes $P_{NDE} = P(X \geq N_{DE} \mid H_0)$





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(ii) the abnormal perturbation of the pathway

- the perturbation for each gene in the pathway is defined as
  $PF(g_i) = \Delta E(g_i) + \sum_{j=1}^{p} \beta_{ij} \frac{PF(g_j)}{N_{DS}(g_j)}$

## The SPIA Methodology

SPIA combines two types of evidence

(ii) the abnormal perturbation of the pathway

- the perturbation for each gene in the pathway is defined as
  $PF(g_i) = \Delta E(g_i) + \sum_{j=1}^{p} \beta_{ij} \frac{PF(g_j)}{N_{DS}(g_j)}$
  - $PF(g_i)$ is the perturbation factor of gene $i$ (not known)
  - $\beta_{ij}$ is the magnitude of effect of gene $j$ on gene $i$; currently, $beta_{ij} = 1$ if $j \to i$
  - $\Delta E(g_i)$ is the fold change in expression of gene $i$
  - $N_{DS}(g_j)$ is the number of downstream genes from gene $j$





## The SPIA Methodology

# The SPIA Methodology

- The accumulated activity of each gene can then be calculated as $ACC(g_i) = B \cdot (I - B)^{-1} \Delta E$



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
**SPIA**
NetGSA
A Systematic Comparison

# The SPIA Methodology

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  - Requires $B$ to be invertible; would not work otherwise

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Introduction
Signal Detection on Networks
Topology-Based Pathway Enrichment Analysis
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
NetGSA
A Systematic Comparison

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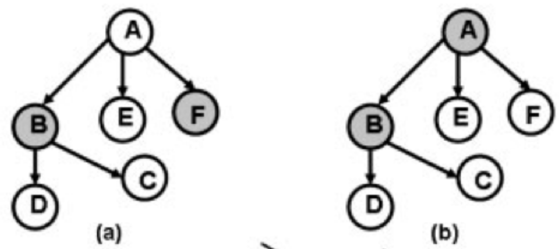
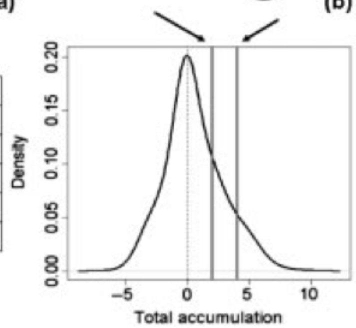
- The p-value for pathway perturbation is given by $P_{PERT} = P(T_A \geq t_A \mid H_0)$, which is calculated using a bootstrap approach

# The SPIA Methodology




| Gene | ΔE | PF | Acc |
|---|---|---|---|
| A | 0 | 0 | 0 |
| B | 2 | 2 | 0 |
| C | 0 | 1 | 1 |
| D | 0 | 1 | 1 |
| E | 0 | 0 | 0 |
| F | 4 | 4 | 0 |
| Total | | | 2.0 |

$P_{PERT}$=0.57




| Gene | ΔE | PF | Acc |
|---|---|---|---|
| A | 1.5 | 1.5 | 0 |
| B | 2 | 2.5 | 0.5 |
| C | 0 | 1.25 | 1.25 |
| D | 0 | 1.25 | 1.25 |
| E | 0 | 0.5 | 0.5 |
| F | 0 | 0.5 | 0.5 |
| Total | | | 4.0 |

$P_{PERT}$=0.24





# The SPIA Methodology

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SPIA combines two types of evidence



Introduction
Signal Detection on Networks
Topology-Based Pathway Enrichment Analysis
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
NetGSA
A Systematic Comparison

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- The final p-value for each pathway is calculated based on the p-values from parts (i) and (ii):
  - $P_G(i) = c_i - c_i \ln(c_i)$
  - $c_i = P_{NDE}(i) P_{PERT}(i)$



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules

PathNet
topologyGSA
**SPIA**
NetGSA
A Systematic Comparison

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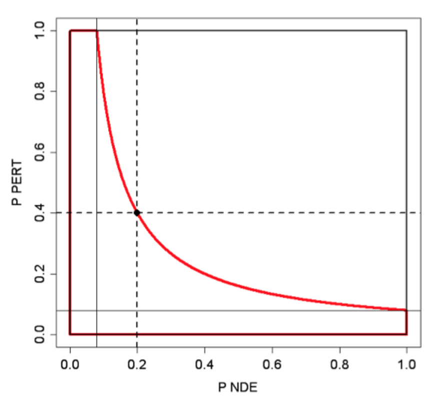


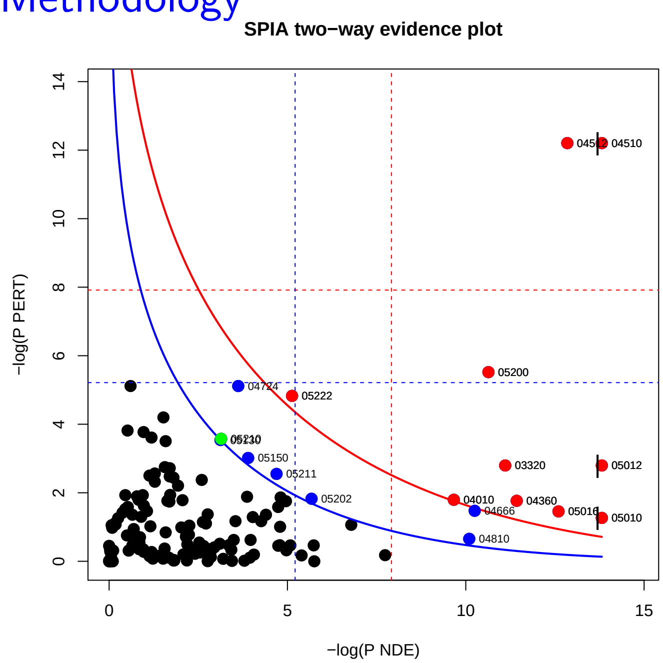
# An Example in R: Data on Colorectal Cancer

```
data(colorectalcancer)

#pathway analysis using SPIA
#use nB=2000 or higher for more accurate results
#uses older version of KEGG signaling pathways graphs
res <- spia(de=DE_Colorectal, all=ALL_Colorectal, organism="hsa", beta=NULL,
    nB=2000, plots=FALSE, verbose=TRUE, combine="fisher")

#now combine pNDE and pPERT using the normal inversion method without
#running spia function again
res$pG=combfunc(res$pNDE,res$pPERT,combine="norminv")
res$pGFdr=p.adjust(res$pG,"fdr")
res$pGFWER=p.adjust(res$pG,"bonferroni")
plotP(res,threshold=0.05)

#highlight the colorectal cancer pathway in green
points(I(-log(pPERT))~I(-log(pNDE)),data=res[res$ID=="05210",],col="green",
    pch=19,cex=1.5)
```





# The SPIA Methodology



## Network-Based Gene Set Analysis (NetGSA)[4]

- Generalizes SPIA, to allow for more complex experiments & incorporate interactions among pathways
- Assesses the overall behavior of arbitrary subnetworks (pathways): changes in gene expression & network structure
- Uses latent variables to model the interaction between genes defined by the the network
- Uses mixed linear models for inference in complex data
- Computationally challenging for large networks, unless pathways separately analyzed (similar to SPIA)

[4] S & Michailidis (2009, 2010); Ma, S & Michailidis (2016)

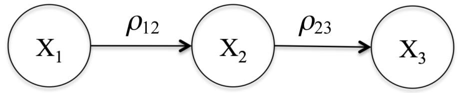




## Problem Setup

- Gene (protein/metabolite) expression data for $K$ experimental conditions and $J_k$ time points
- Network information (partially) available in the form of a directed weighted graph $G = (V, E)$, with vertex set $V$ corresponding to the genes/proteins/metabolites and edge set $E$ capturing their associations
- Network edges can be directed $j \to k$ or undirected $j \leftrightarrow k$
- Edges defines the effect of nodes on their immediate neighbors; the weight associated with each edge corresponds to the value of partial correlation
- Represent the network by its adjacency matrix $A$: $A_{jk} \neq 0$ iff $k \to j$ & for undirected edges, $A_{jk} = A_{kj}$

# The Latent Variable Model: Main Idea

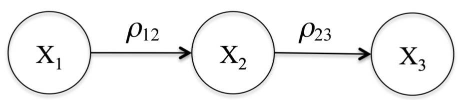



$$
\begin{aligned}
X_1 &= \gamma_1 \\
X_2 &= \rho_{12}X_1 + \gamma_2 = \rho_{12}\gamma_1 + \gamma_2 \\
X_3 &= \rho_{23}X_2 + \gamma_3 = \rho_{23}\rho_{12}\gamma_1 + \rho_{23}\gamma_2 + \gamma_3
\end{aligned}
$$



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
**NetGSA**
A Systematic Comparison

# The Latent Variable Model: Main Idea

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\end{aligned}
$$

Thus $X = \Lambda\gamma$ where

$$
\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & 1 & 0 \\ \rho_{12}\rho_{23} & \rho_{23} & 1 \end{pmatrix}
$$

# The Latent Variable Model

- Let $Y$ be the $i$th sample in the expression data
- Let $Y = X + \varepsilon$, with signal $X$ and noise $\varepsilon \sim N_p(0, \sigma_\varepsilon^2 I_p)$
- The influence matrix $\Lambda$ measures the propagated effect of genes on each other through the network, and can be calculated based on the adjacency matrix $A$
- Using $X = \Lambda\gamma$, we get

$$Y = \Lambda\gamma + \varepsilon, \quad \Rightarrow \quad Y \sim N_p(\Lambda\mu, \sigma_\gamma^2 \Lambda\Lambda' + \sigma_\varepsilon^2 I_p)$$

where $\gamma \sim N_p(\mu, \sigma_\gamma^2 I_p)$ are latent variables





# Mixed Linear Model Representation

Rearranging the expression matrix into $np$-vector $\mathbf{Y}$, we can write

$$\mathbf{Y} = \mathbf{\Psi}\boldsymbol{\beta} + \mathbf{\Pi}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are fixed and random effect parameters and

$$\boldsymbol{\varepsilon} \sim N_{np}(\mathbf{0}, R(\theta_\varepsilon)), \qquad \boldsymbol{\gamma} \sim N_{np}(\mathbf{0}, \sigma_\gamma^2 \mathbf{I_{np}})$$

- Temporal Correlation incorporated through $R$

# Mixed Linear Model Representation

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- Temporal Correlation incorporated through $R$

In general, the design matrices, $\mathbf{\Psi}$ and $\mathbf{\Pi}$ depend on the experimental settings (similar to ANOVA), and are functions of $\Lambda$





# Estimation of MLM Parameters

MLE for $\beta$:

$$\hat{\beta} = \left(\Psi' \hat{W}^{-1} \Psi\right)^{-1} \Psi' \hat{W}^{-1} \mathbf{Y}$$

where $W = \sigma_\gamma^2 \Pi\Pi' + R$.

$\hat{\beta}$ depends on estimates of $\sigma_\gamma^2$ and $\theta_\varepsilon^2$ (estimated using restricted maximum likelihood (REML)).

## Inference using MLM

- Let $\ell$ be a contrast vector (a linear combination of fixed effects), and consider the test:

$$H_0 : \ell\beta = 0 \quad vs. \quad H_1 : \ell\beta \neq 0$$



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
**NetGSA**
A Systematic Comparison

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- Let $\ell$ be a contrast vector (a linear combination of fixed effects), and consider the test:

$$H_0 : \ell\beta = 0 \quad vs. \quad H_1 : \ell\beta \neq 0$$

- Use t-test to test the significance of each hypothesis separately

$$T = \frac{\ell\hat{\beta}}{\sqrt{\ell\hat{C}\ell'}}$$

where $C = (\Psi' W^{-1} \Psi)^{-1}$

- Under the null hypothesis, $T$ is approximately $t$-distributed with degrees of freedom that needs to be estimated

## "Optimal" Choice of Contrast Vector

- An intuitive choice is the indicator (membership) vector for the pathway, **b**, but this only captures changes in mean
- Need to *de-couple the effect of subnetwork* from other nodes



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
**NetGSA**
A Systematic Comparison

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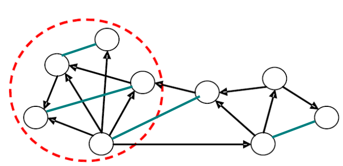

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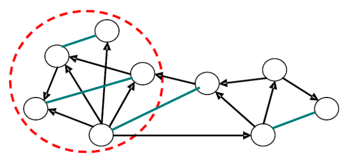


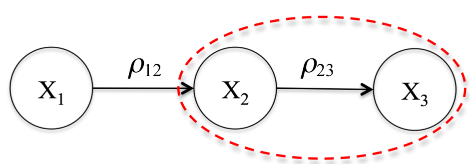
- Can be shown that $(\mathbf{b}\Lambda \cdot \mathbf{b})\gamma$ is not affected by nodes outside $\mathbf{b}$, but includes the effects of nodes in $\mathbf{b}$ on each other
- In the case-control case, the optimal contrast vector is:

$$\ell^* = \left(-\mathbf{b} \cdot \mathbf{b}\Lambda^C, \mathbf{b} \cdot \mathbf{b}\Lambda^T\right)$$





## "Optimal" Choice of Contrast Vector




$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & 1 & 0 \\ \rho_{12}\rho_{23} & \rho_{23} & 1 \end{pmatrix}$$

Consider the set, $\mathbf{b} = (0, 1, 1)$; then

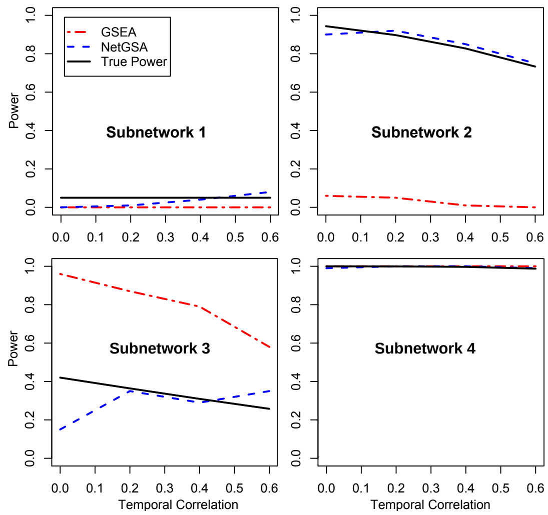
$$(\mathbf{b}\Lambda) = (\rho_{12} + \rho_{12}\rho_{23}, 1 + \rho_{23}, 1)$$

On the other hand,

$$(\mathbf{b}\Lambda \cdot \mathbf{b}) = (0, 1 + \rho_{23}, 1)$$

## Comparison in Simulated Data

| Subnetwork | Mean | Network Influence |
|---|---|---|
| 1 | $\mu_1 = \mu_2 = 1$ | $\rho_1 = \rho_2 = 0.2$ |
| 2 | $\mu_1 = 1, \mu_2 = 2$ | $\rho_1 = \rho_2 = 0.2$ |
| 3 | $\mu_1 = \mu_2 = 1$ | $\rho_1 = 0.2, \rho_2 = 0.7$ |
| 4 | $\mu_1 = 1, \mu_2 = 2$ | $\rho_1 = 0.2, \rho_2 = 0.7$ |








## Yeast Galactose Utilization Pathway

*Ideker et al* (2001) data on yeast Galactose Utilization Pathway

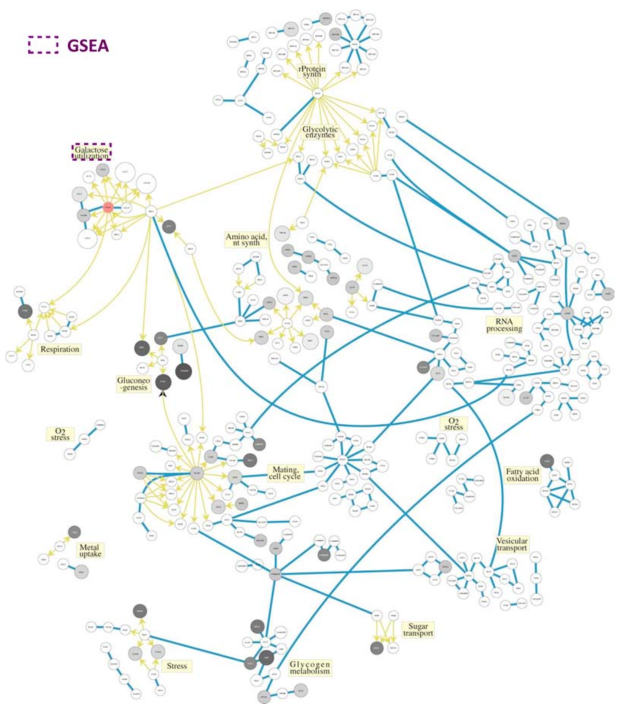
- Gene expression data for 2 experimental conditions: (gal+) and (gal–)
- Gene-gene and protein-gene interactions as well as association weights found from previous studies
- Q: which pathways respond to the change in growth medium?

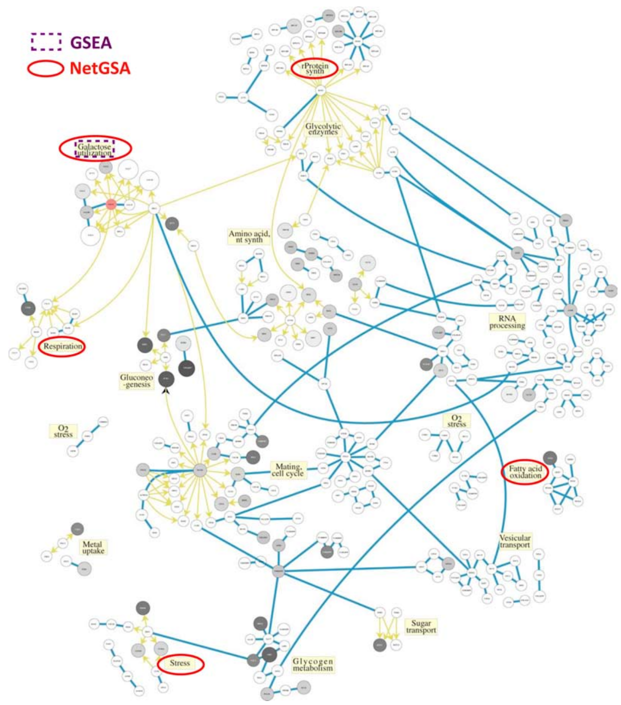
# Analysis of Yeast GAL Data

- Data:
  - gene expression data for 343 genes
  - 419 interactions found from previous studies and integration with protein expression (association among genes also available)
- Results:
  - GSEA finds *Galactose Utilization Pathway* significant
  - NetGSA finds several other pathways with biologically meaningful functions related to survival of yeast cells in gal–














## Environmental Stress Response in Yeast

Gene expression data on Yeast Environmental Stress Response (ESR) (*Gasch et al.*, 2000)

- 3 combinations of experimental factor, heat shock and osmotic changes (sorbitol), over 3 time points
- Temporal correlation
- Network correlation
- Q: Which pathways indicate response to environmental stress
  - in different experimental conditions
  - over time

# Yeast ESR Data

## Gasch et al (2000)

- Gene Expression Data

| Experiment | Obs. Time (after 33C) |
|---|---|
| Mild heat shock (*29C to 33C*), no sorbitol | 5, 15, 30 min |
| Mild Heat Shock, 1M sorbitol at 29C & 33C | 5, 15, 30 min |
| Mild Heat Shock, 1M sorbitol at 29C | 5, 15, 30 min |

- Network Data
  - Use YeastNet (*Lee et al.*, 2007) for gene-gene interactions (102,000 interactions among 5,900 yeast genes)
  - Use independent experiments of *Gasch et al.* to estimate weights
  - Pathways are defined using GO functions





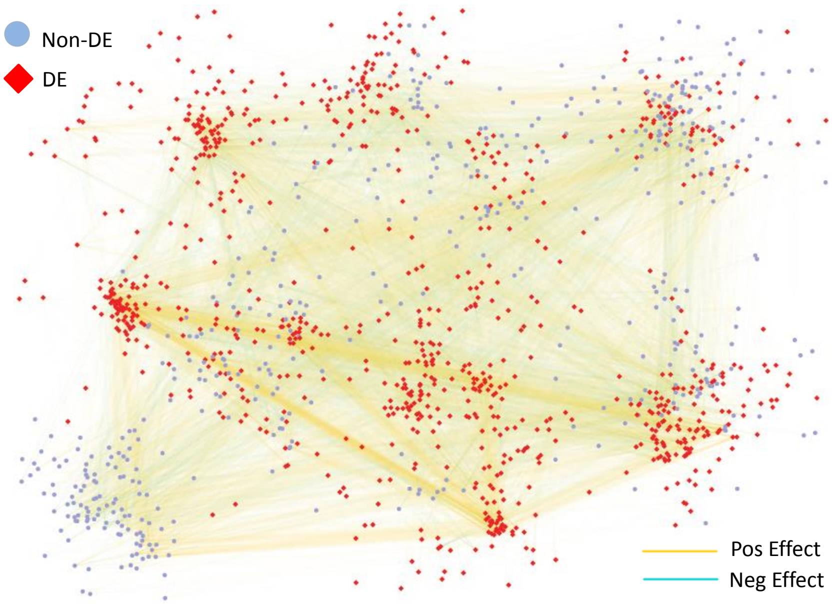
# Model and Results

- Model: Let $j$ and $k$ be indices for time and levels of sorbitol

$$\mathbb{E}Y_{11} = \Lambda\mu, \qquad \mathbb{E}Y_{jk} = \Lambda(\mu + \alpha_j + \delta_k) \quad j, k = 2, 3$$

- Temporal correlation is modeled directly via $R$ (as $AR(1)$ process)
- Results:
  - $\sim 3000$ genes,
  - 47 pathways showed significant changes of expression
  - 24 pathways showed changes over time
  - 29 pathways showed changes in response to different sorbitol levels
  - 12 pathways showed both types of changes
  - Significant pathways overlap with gene functions from *Gasch et al.*

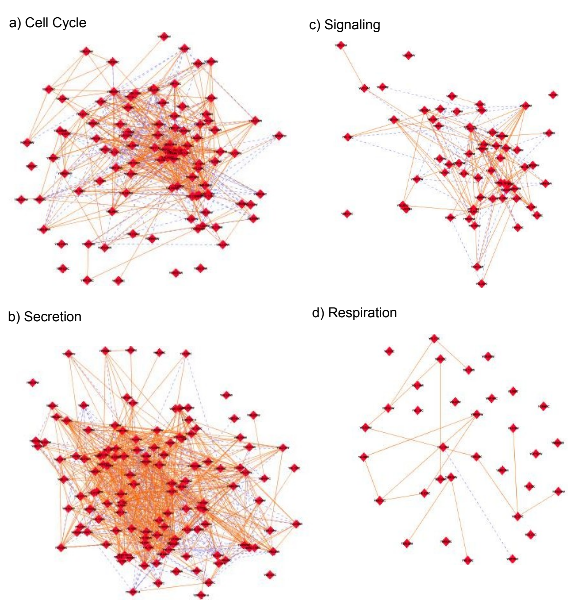
# Yeast ESR Network





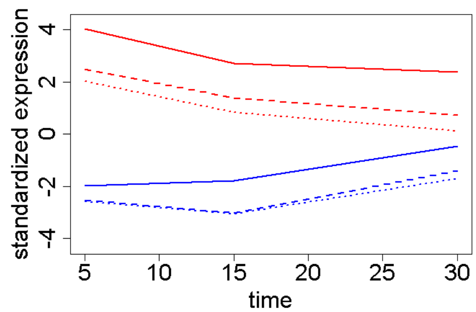



# Significant subnetworks



## Expression Profiles

### Average Standardized Expression Levels of Pathways




- Induced and Suppressed Pathways
- Can observe the transient patterns of expressions as predicted by *Gasch et al.*





## Effect of Noise In Network Information

- Let $\tilde{A}$ be observed network information, and $A$ be the truth.
- It can be shown that, if $\|\tilde{A} - A\|$ is small then, NetGSA still works (is *asymptotically most powerful unbiased test*)

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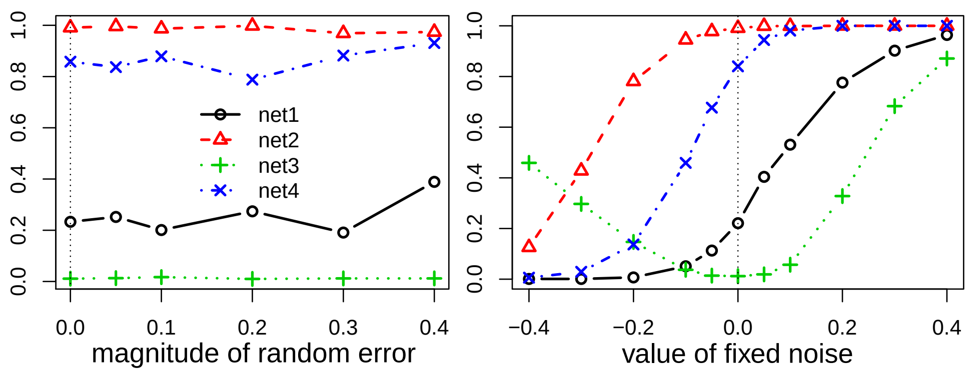


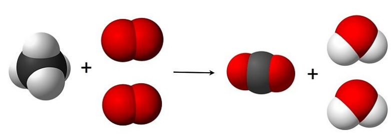





# Metabolic Profiling in Bladder Cancer

Targeted metabolic profiling of bladder cancer (BCa)[5]

- 58 bladder cancer and adjacent benign samples
- Pathways information obtained from KEGG

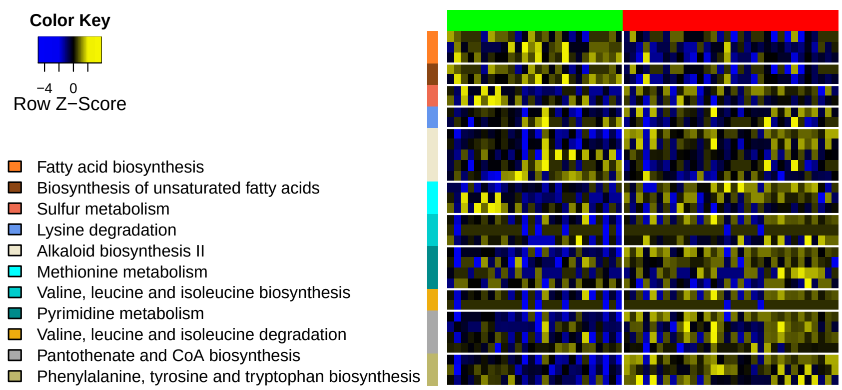



- Varying number of identified metabolites per pathway (3-15)
- Q: Which pathways show differential activity in BCa?

[5]Putluri et al. (2012)

# Metabolic Profiling in BCa

- 63 metabolites identified, mapped to 70 pathways
- 27 pathways with at least 3 members

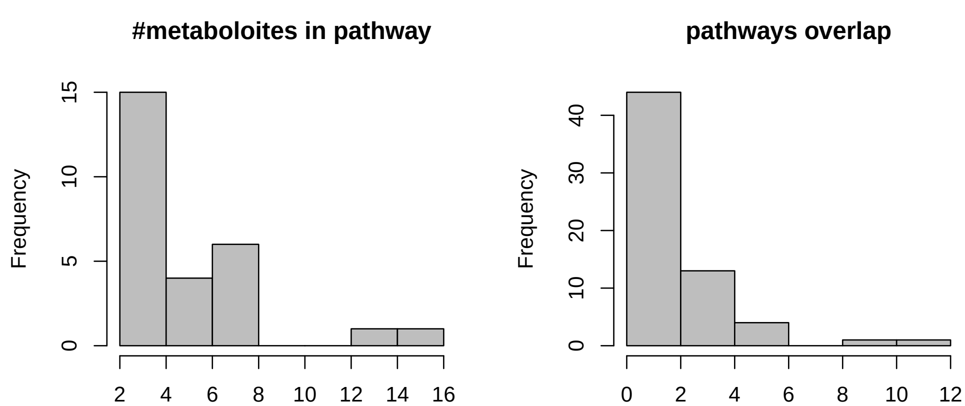





Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
**NetGSA**
A Systematic Comparison

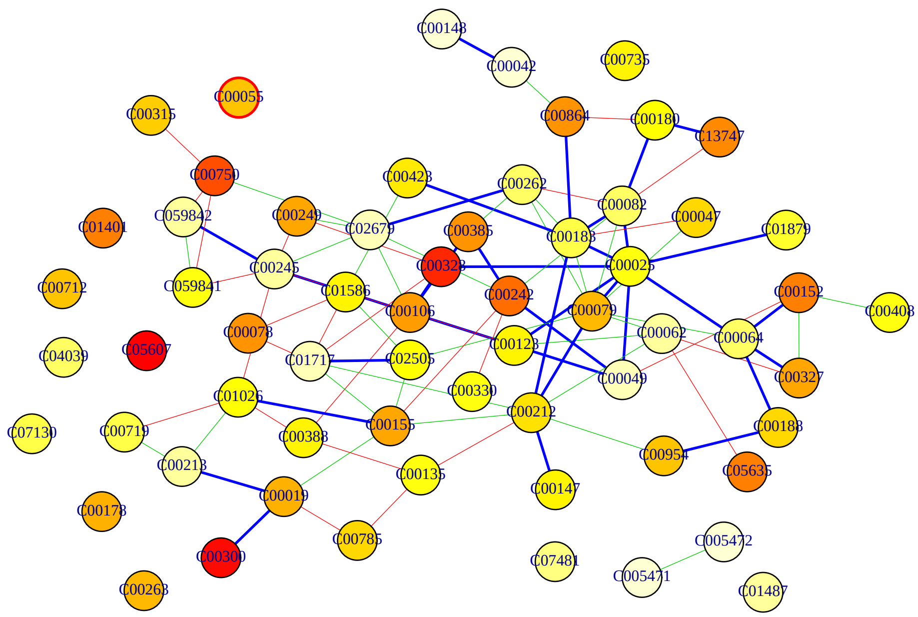
# Metabolic Profiling in BCa

- Small pathway sizes & significant overlap among pathways




- Existing methods may not work well...

# Metabolic Interaction Network








# Significant Pathways

- GSEA does not identify any pathway as differential
- GSA identifies Fatty Acid Biosynthesis as differential
- NetGSA identifies another 7 pathways corresponding to role of Amino Acid Metabolism in BCa, similar to *Putluri et al* (2012)

# R-Package `netgsa`

- Basic usage:

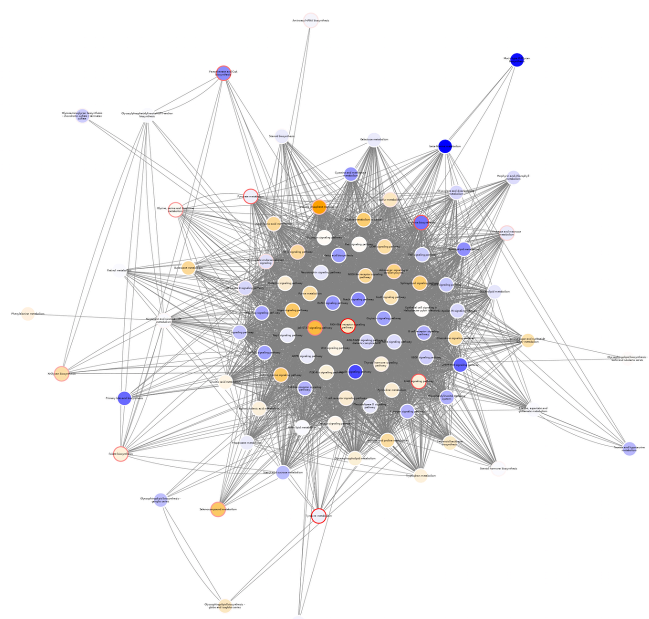
  `NetGSA(A, x, group, pathways)`

- `A`: List of $p \times p$ weighted adjacency matrices for each condition (e.g. normal vs cancer), to capture changes in the network
- `pathways`: a $K \times p$ 0-1 matrix of pathway membership: $\text{pathways}_{k,j} = 1$ if gene/.../metabolite $j$ in pathway $k$
- Output: test statistics and p-values for each pathway
- The NetGSA function takes a weighted `A` as input. The package includes functions to learn `A` for undirected networks from a (partial) list of network edges

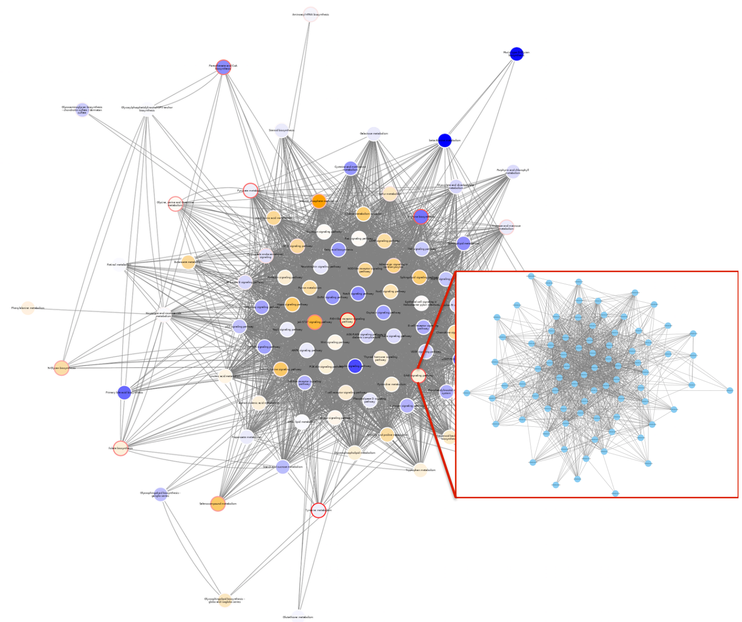




# R-Package `netgsa`



## R-Package `netgsa`



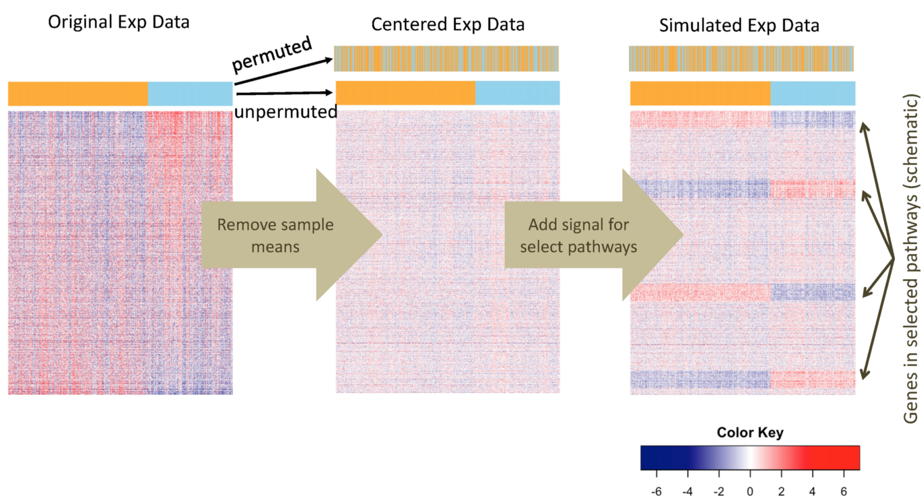




## Comparison Using Synthetic Data (Ma, S., Michailidis, 2019)

- Comparison of topology-based pathway enrichment methods using two synthetic data sets
  - Gene expression data $p \approx 3000$
  - Metabolomics data $p \approx 100$
- *In silico* data sets with known signal:
  1. Remove the original signal, but keep the correlation structure
  2. Perturb means in one condition (differential expression) for nodes in selected pathways
  3. Also use sample permutation to create data with equal correlation structure

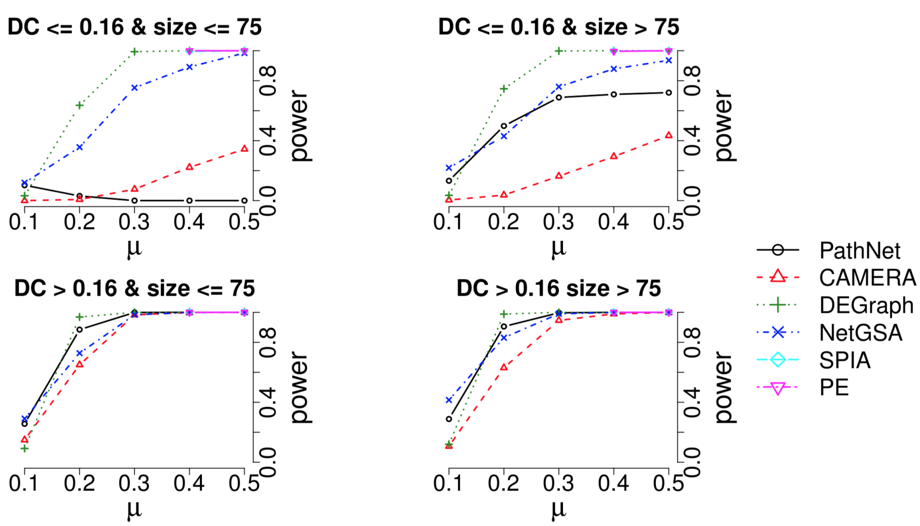
# Comparison Using Synthetic Data



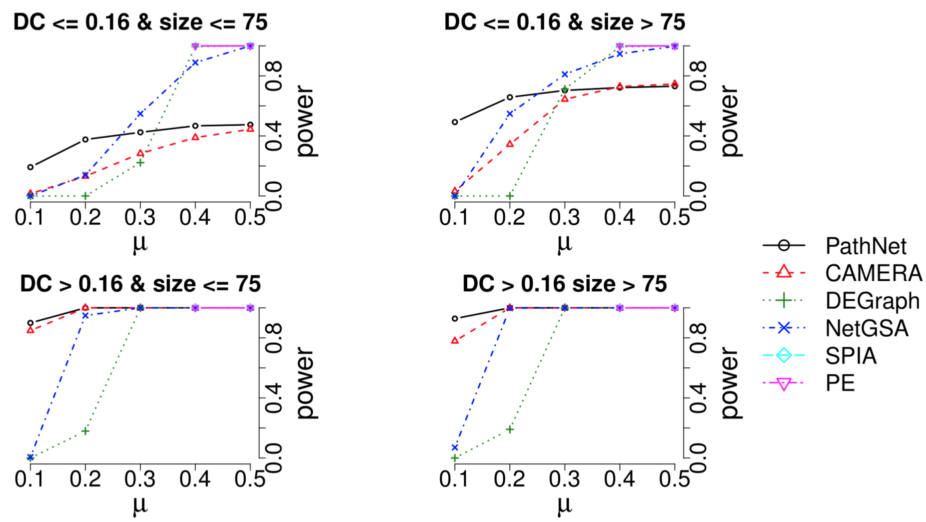





# Results for Gene Expression Data — Equal Covariance



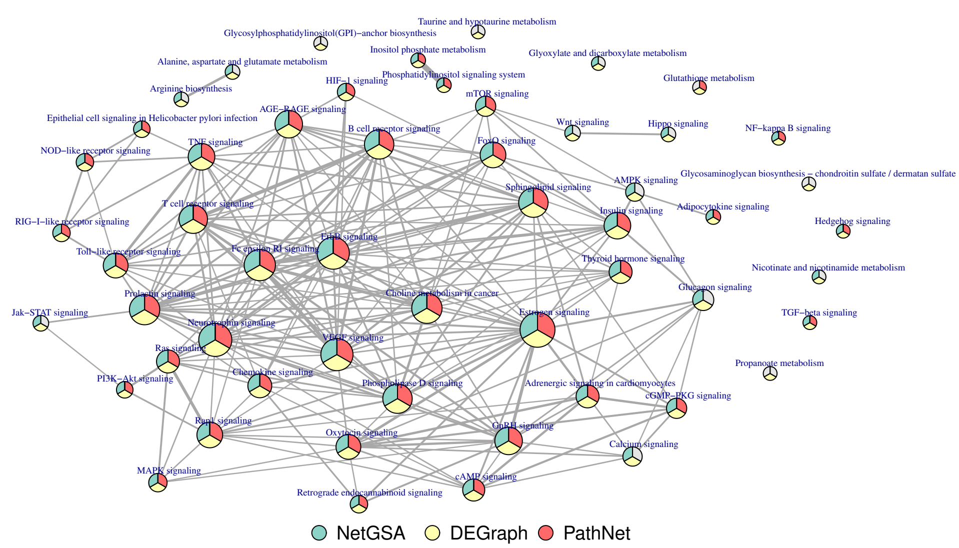
# Results for Gene Expression Data — Diff Covariance



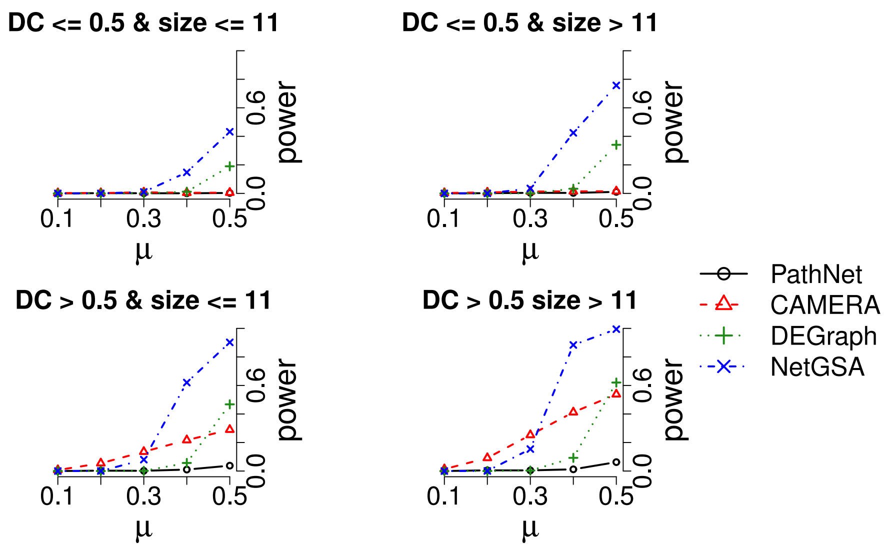





# Results for Gene Expression Data



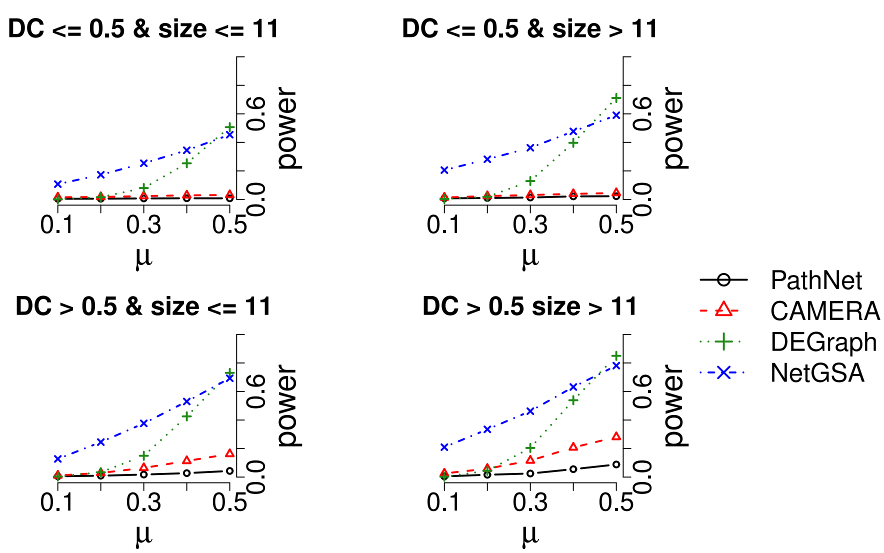
# Results for Metabolomics Data — Equal Covariance



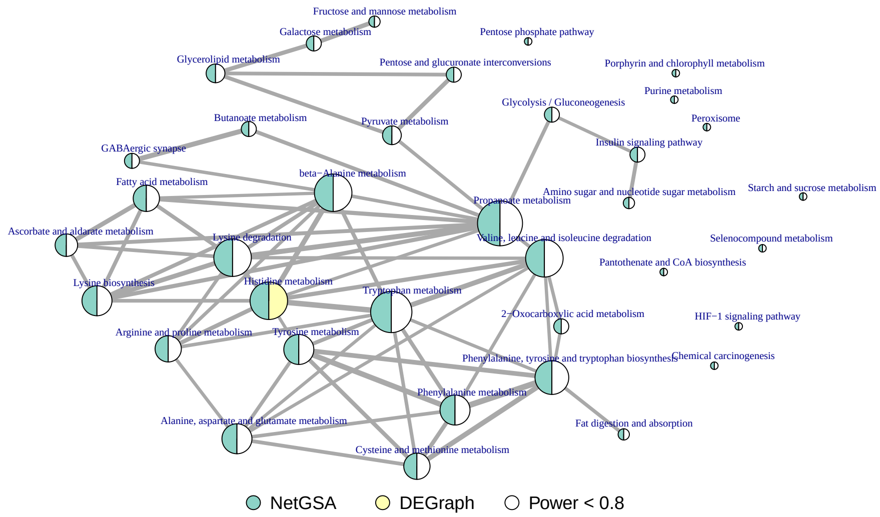





# Results for Metabolomics Data — Diff Covariance



# Results for Metabolomics Data








# Multi-Omics NetGSA

Pan-cancer integration of expression, methylation and CNV in BRAF (TCGA data)[6]

[6] Zhang et al (2018)

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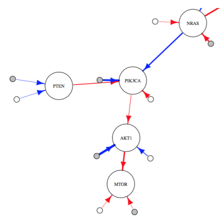

[6] Zhang et al (2018)



Introduction
Signal Detection on Networks
**Topology-Based Pathway Enrichment Analysis**
De-Novo Identification of Enriched Modules
PathNet
topologyGSA
SPIA
NetGSA
**A Systematic Comparison**

# Multi-Omics NetGSA

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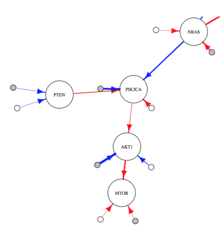

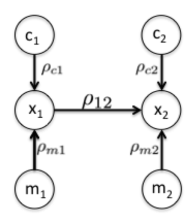


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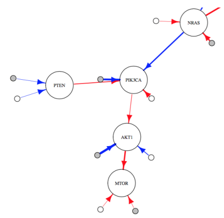

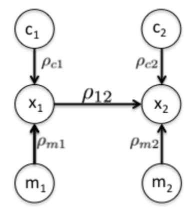


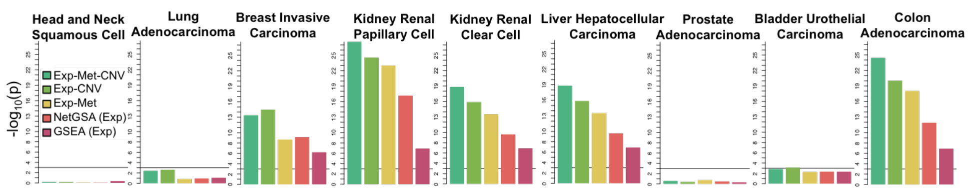


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# Identifying Enriched Modules in Networks

## Identifying Enriched Modules in Networks

Two general strategies:

- Assess the significance of data-driven modules (WGCNA):
  1. Identify modules (network clustering, etc)
  2. Assess the significance of modules
- Search for enriched (connected) subnetworks (often using greedy search methods)



Introduction
Signal Detection on Networks
Topology-Based Pathway Enrichment Analysis
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Walktrap

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- Assess the significance of data-driven modules (WGCNA):
  1. Identify modules (network clustering, etc)
  2. Assess the significance of modules
- Search for enriched (connected) subnetworks (often using greedy search methods)
- Advantage: No need to rely on known pathways — especially useful when known pathways are not complete, etc
- Disadvantage: Interpretation may become challenging...

# WGCNA[7]

- WGCNA is a method for constructing weighted gene co-expression networks (discussed in the next lecture), which also facilitates topology-based enrichment analysis, in a different way than many other topology-based methods

[7] Horvath & Zhang (2005); Langfelder et al (2008)



Introduction
Signal Detection on Networks
Topology-Based Pathway Enrichment Analysis
De-Novo Identification of Enriched Modules
WGCNA
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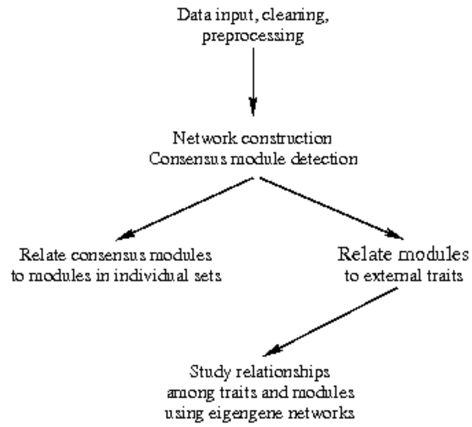
# WGCNA[7]

- WGCNA is a method for constructing weighted gene co-expression networks (discussed in the next lecture), which also facilitates topology-based enrichment analysis, in a different way than many other topology-based methods
- Here's how it works:
  1. Estimate the co-expression network (more in the next lecture)
  2. Find modules by clustering the nodes in the estimated network
  3. Summarize the expressions of genes in each module using PCA (eigen-genes)
  4. Test if the eigen-genes are associated with the outcome

[7] Horvath & Zhang (2005); Langfelder et al (2008)

# WGCNA

- Here's how it works:




Let's look at an example in R...





# WGCNA

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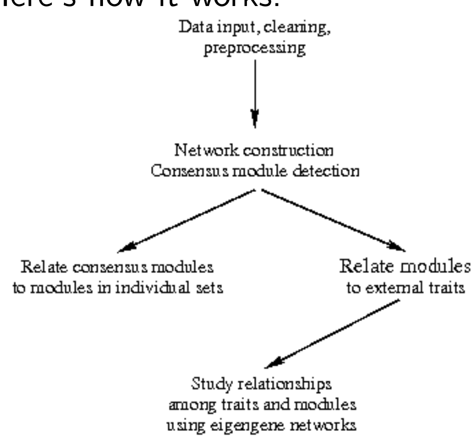
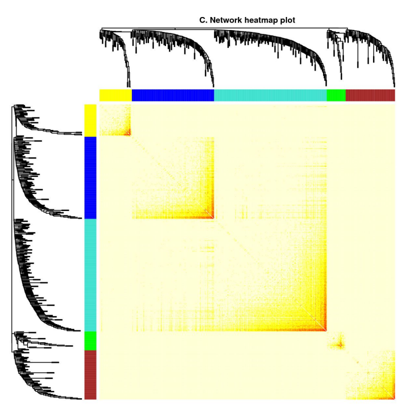



Let's look at an example in R...

# Walktrap[8]

- Searches for connected modules containing significant genes
  - Weights each edges based on the significance of its corresponding nodes

$$w_{ij} = \big(|\mathrm{FC}_i| + |\mathrm{FC}_j|\big)/2$$

  - Connected significant modules are found through community detection using a random walk with transition probability

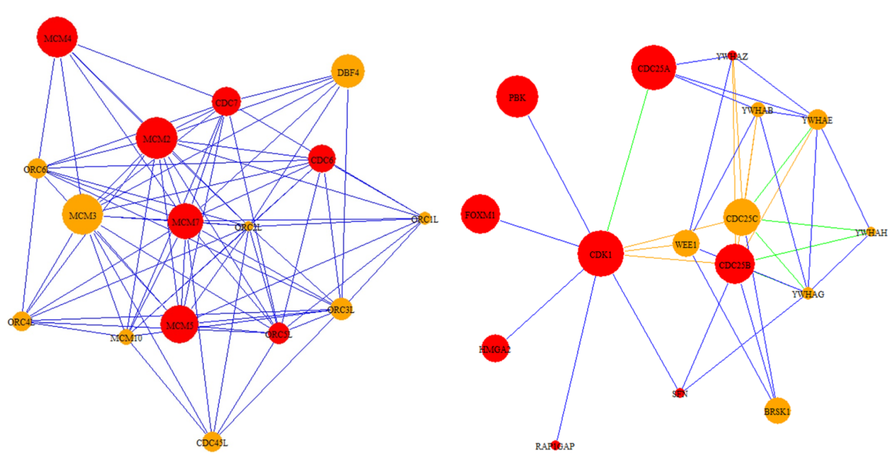
$$P_{ij} = \frac{w_{ij}}{\sum_j w_{ij}}$$

[8] Petrochilos et al (2013)





# Identifying Cancer-Related Modules



# Summary

- Network-based methods (centrality-based, pathway topology, etc) rely on network information — helpful if correct network information avail
- What if network information is not available?





# Summary

- Focus is shifting towards estimating changes in the structure of networks: differential network biology[9]

[9] Ideker & Krogan (2012); S (2021)

# Pathway & Network Analysis of Omics Data: Learning Undirected Networks

Ali Shojaie
Department of Biostatistics
University of Washington
faculty.washington.edu/ashojaie

Summer Institute for Statistical Genetics – 2023

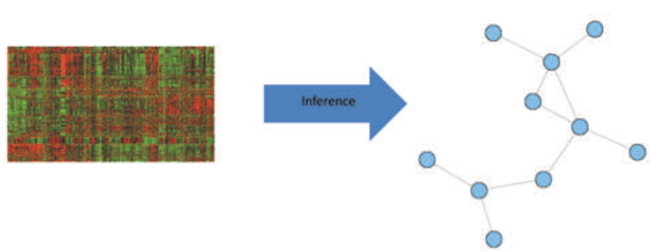




## Learning Undirected Networks

Learn network from data (structure learning):

- Data matrix: $X_{n \times p}$.
- Features correspond to the $p$ nodes in the network.
- Goal: Learn edges between nodes $\equiv$ learn the statistical relationships between features.



## Why Do We Need Network Inference?

- Despite progress, our knowledge of interactions is limited.
- The entire genome is a vast landscape, and experiments for discovering networks are very expensive.
- From a statistical point of view, network estimation is related to estimation of covariance matrices, which has many independent applications in statistical inference and prediction (*more about this later*).
- Finally, and perhaps most importantly, gene and protein networks are dynamic and changes in these networks have been attributed to complex diseases.





## Network Inference — An Overview

## Network Inference — An Overview

Two general classes of network inference methods:

- Methods based on marginal measures of association:
  - Co-expression Networks (based on linear measures of association)
  - Methods based on mutual information (can accommodate non-linear associations)
- Methods based on conditional measures of association:
  - Methods assuming (multivariate) normality (`glasso`, etc)
  - Generalizations to allow for nonlinear dependencies (`nonparanormal`, etc)





## Graphical Models

Probabilistic Graphical Models [1]

Joint multivariate probability distribution where dependencies can be represented as a network.

Advantages:

- Graphical models offer efficient factorized forms for joint distributions with easily interpretable dependencies.
  - **Conditional dependencies** denoted via an edge in network.
- Convenient visual representation.

[1] For a detailed introduction see *Graphical Models, Exponential Families, and Variational Inference*; Wainwright & Jordan (2008)

# Marginal Association Networks





## Correlation Networks (Association Networks)

- Simplest (and most-widely used!) method for estimating networks — key assumption:
  large correlation $\equiv$ presence of an edge
- Let $r(i,j)$ be correlation between $X_i$ and $X_j$; we claim an edge between $i$ and $j$ if $|r(i,j)| > \tau$.
  - $\tau$: a user-specified threshold (tuning parameter).

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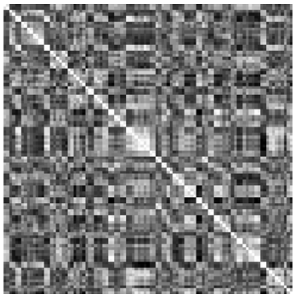
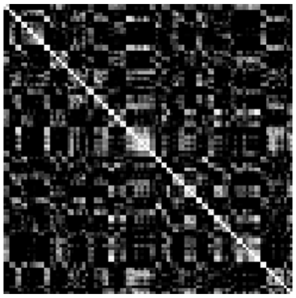


Correlation matrix



Thresholded correlation matrix





# Limitations of Correlation Networks

1. The estimation is highly dependent on the choice of $\tau$.
2. Correlations capture **linear** associations, but many real-world relationships are nonlinear.
3. Large correlations can occur due to confounding.

# Limitations of Correlation Networks

The estimation is highly dependent on the choice of $\tau$.



Introduction
**Marginal Association Networks**
Conditional Independence Graphs

# Limitations of Correlation Networks

The estimation is highly dependent on the choice of $\tau$.

- We can work with weighted co-expression networks (WGCNA)
- We can instead test $H_0 : r_{xy} = 0$
  - A commonly used test is based on the Fisher transformation

$$Z = \frac{1}{2} \ln \left( \frac{1+r}{1-r} \right) = \mathrm{artanh}(r) \sim_{H_0} N\left(0, \frac{1}{\sqrt{n-3}}\right)$$

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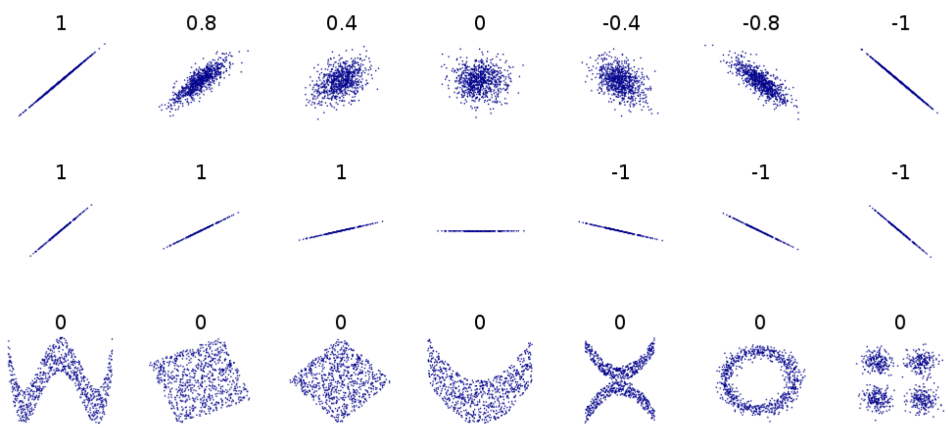

# Limitations of Correlation Networks

Correlations capture **linear** associations, but many real-world relationships are nonlinear.



Introduction
Marginal Association Networks
Conditional Independence Graphs

# Limitations of Correlation Networks

Correlations capture **linear** associations, but many real-world relationships are nonlinear.

- We can use other measures of association, for instance, Spearman correlation or Kendal's $\tau$.
  - These methods define the correlation between two variables, based on the ranking of observations, and not their exact values.
  - They can better capture non-linear associations.
- We can instead use mutual information; this has been used in many algorithms, e.g. ARACNE.

## ARACNE: Algorithm for the Reconstruction of Accurate Cellular NEtworks[2]

[2] Margolin et al (2006)



Introduction
Marginal Association Networks
Conditional Independence Graphs

## ARACNE: Algorithm for the Reconstruction of Accurate Cellular NEtworks[2]

1. Identifies statistically significant gene-gene co-regulation based on mutual information

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# ARACNE: Algorithm for the Reconstruction of Accurate Cellular NEtworks[2]

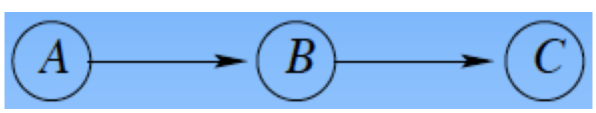
1. Identifies statistically significant gene-gene co-regulation based on mutual information
2. It then eliminates indirect relationships in which two genes are co-regulated through one or more intermediates

[2] Margolin et al (2006)

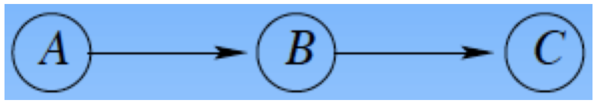




# Key Idea: Data Processing Inequality (DPI)



# Key Idea: Data Processing Inequality (DPI)




$$I(A, C) \leq min[I(A, B), I(B, C)]$$

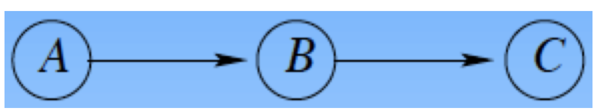
where

$$I(g_i, g_j) = \log P(g_i, g_j)/P(g_i)P(g_j)$$





Key Idea: Data Processing Inequality (DPI)




$$I(A, C) \leq min[I(A, B), I(B, C)]$$

where

$$I(g_i, g_j) = \log P(g_i, g_j)/P(g_i)P(g_j)$$

- Look at every triplet and remove the weakest link
- Need to estimate marginal and joint (pairwise) probabilities (using Gaussian Kernel)

# Algorithm Details



Introduction
Marginal Association Networks
Conditional Independence Graphs

# Algorithm Details

- The algorithm examines each gene triplet for which all pairwise MIs are greater than a cut-off and removes the edge with the smallest value based on DPI.

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# Algorithm Details

- The algorithm examines each gene triplet for which all pairwise MIs are greater than a cut-off and removes the edge with the smallest value based on DPI.
  - Each triplet is analyzed even if its edges have been selected for removal by prior DPI applications to other triplets.
  - The least of the three MIs can come from indirect interactions only, and checking against the DPI may identify gene pairs that are not independent, but still do not interact.

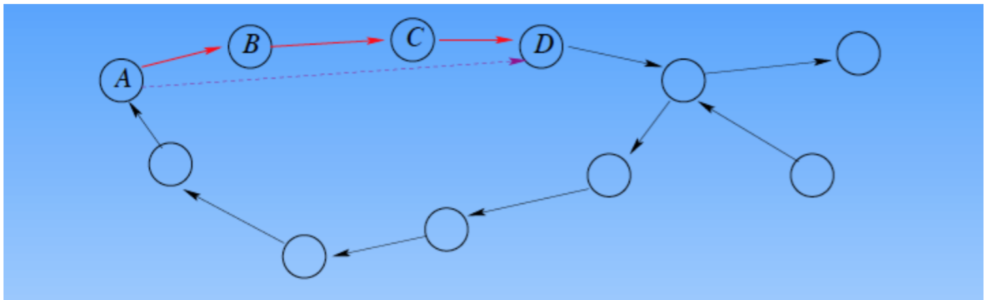




# Rationale and Guarantees

- If MIs are estimated with no errors, then ARACNE reconstructs the underlying interaction network exactly, if the network is a tree and has only pairwise interactions.
- The maximum MI spanning tree is a subnetwork of the network built by ARACNE.

# Rationale and Guarantees

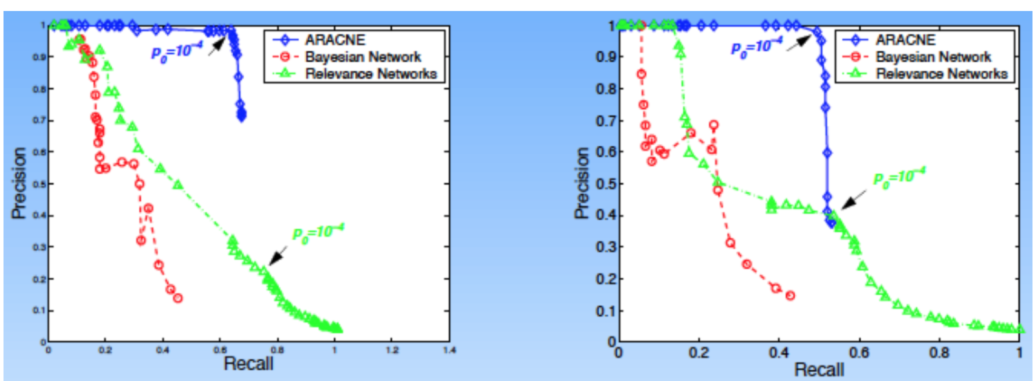



Theorem. Let $\pi_{ik}$ be the set of nodes forming the shortest path in the network between nodes $i$ and $k$. Then, if MIs can be estimated without errors, ARACNE reconstructs an interaction network without false positives edges, provided: (a) the network consists only of pairwise interactions, (b) for each $j \in \pi_{ik}$, $I_{ij} \geq I_{ik}$. Further, ARACNE does not produce any false negatives, and the network reconstruction is exact iff (c) for each directly connected pair $ij$ and for any other node $k$, we have $I_{ij} > \min[I_{ik}, I_{jk}]$.

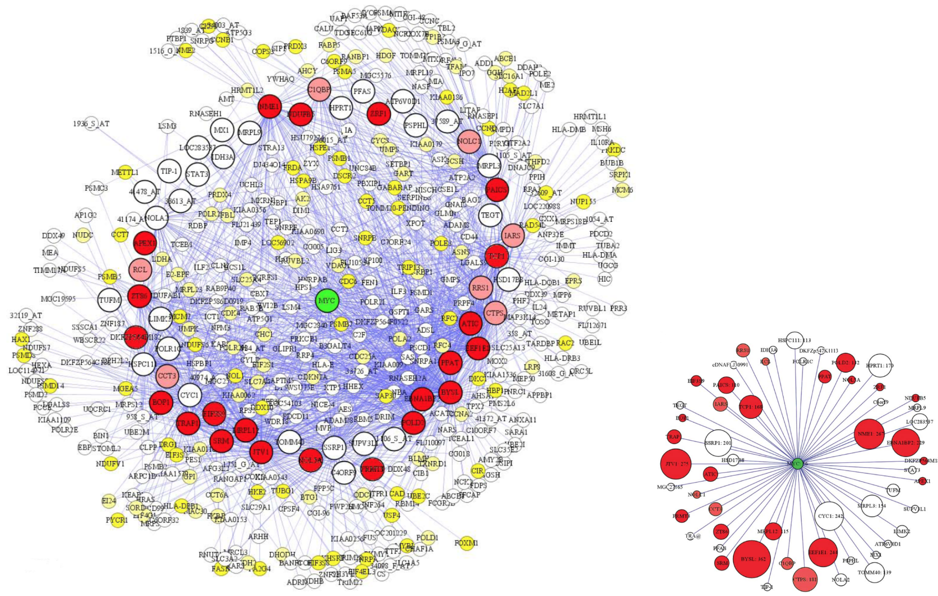




# Performance on Synthetic Data



# Application: B-lymphocytes Expression Data







# Application: B-lymphocytes Expression Data

- MYC (proto-oncogene) subnetwork (2063 genes)
- 29 of the 56 (51.8%) predicted first neighbors biochemically validated as targets of the MYC transcription factor.
- New candidate targets identified, 12 experimentally validated.
  - 11 proved to be true targets.
- Candidate targets not validated can possibly be correct too.

## Software

- Implemented in the R-package `minet`:

```
source("http://bioconductor.org/biocLite.R")
biocLite("minet")
```

- Main estimation function `aracne(mim, eps=0)`
  - mim: mutual information matrix

    ```
    mim <- build.mim(syn.data, estimator="spearman")
    ```
  - eps: threshold for setting an edge to zero, prior to searching over triplets





## Limitations of Correlation Networks

Large correlations can occur due to confounding.

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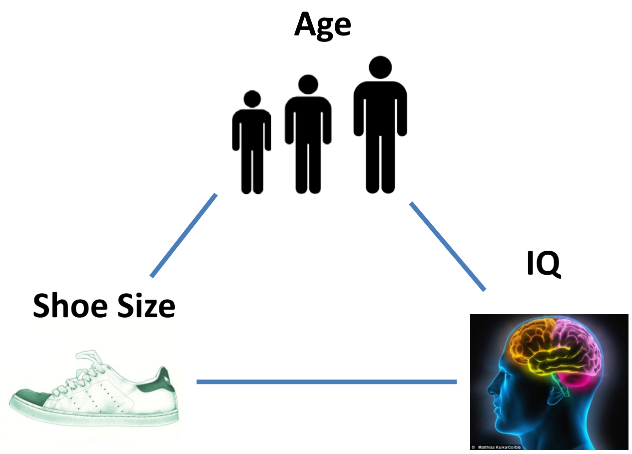


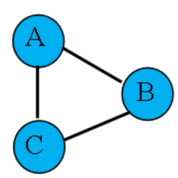




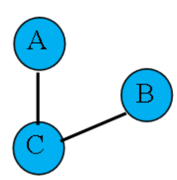

# Markov Networks

Markov network
An *undirected graphical model* that characterizes conditional dependence ($\equiv$ direct relationships).

- *Edge*: Two nodes are **conditionally dependent**.
- *No edge*: Two nodes are **conditionally independent**.
- Conditions on all other nodes.




$$A \perp B \mid C$$



## Markov Networks — Conditional Dependence

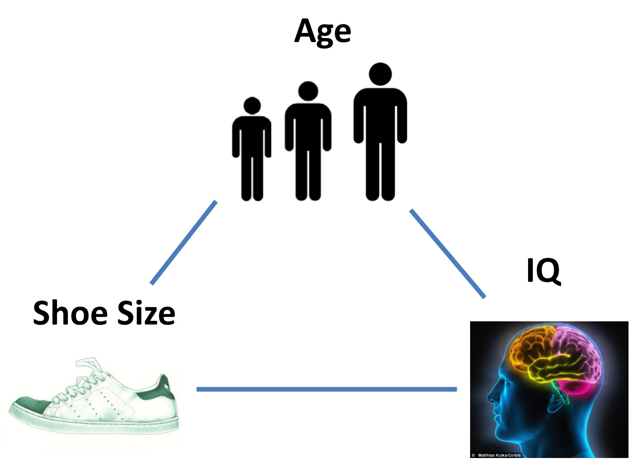
Regression Interpretation:

- Imagine trying to predict the observations in Node A (response) by the observations of all other nodes (predictors).
- Node B predictive of Node A (with all other nodes in model).
  - A is conditionally dependent on B.
  - Edge.
- Because of other nodes in model, Node B does not add any predictive value for Node A.
  - A is conditionally independent of B.
  - No Edge.



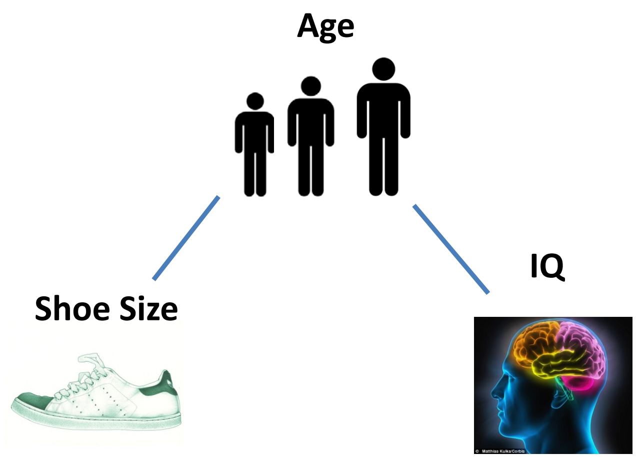


## Markov Networks — Conditional Dependence




Correlation.

## Markov Networks — Conditional Dependence




Conditional Dependence.





## Markov Networks — Conditional Dependence

How can we learn conditional dependencies?

- $A$ and $B$ are conditionally independent given $C$ if

$$P(A, B \mid C) = P(A \mid C)P(B \mid C)$$

  - Generally difficult (need to estimate multivariate densities).
- Alternatively, can use nonparametric approaches, e.g. conditional mutual information – not easy in high dimensions.
- Often resort to models, or simple measures, such as partial correlations...

# Partial Correlation

- Partial correlation measures the correlation between $A$ and $B$ after the effect of the other variables are removed.
  - In our example, this means correlation between shoe size and IQ, after adjusting for age.





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- The partial correlation between $A$ and $B$ given $C$ is given by:

$$\rho_{AB\cdot C} \equiv \rho(A,B|C) = \frac{\rho_{AB} - \rho_{AC}\rho_{BC}}{\sqrt{1-\rho_{AC}^2}\sqrt{1-\rho_{BC}^2}}.$$

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- Alternatively, regress $A$ on $C$ and get the residual, $r_A$; do the same for $B$ to get $r_B$. The partial correlation between $A$ and $B$ give $C$ is $\mathrm{Cor}(r_A, r_B)$.





## Partial Correlation

- Partial correlation is symmetric $\Rightarrow$ undirected network
- Partial correlation takes values between -1 and 1
- In partial correlation networks, we draw an edge between $A$ and $B$, if the partial correlation between them is large
- Calculation of partial correlation is more involved

# A Simple Example

$$\textit{Correlation} = \begin{bmatrix} 1 & .8 & .7 \\ .8 & 1 & .8 \\ .7 & .8 & 1 \end{bmatrix} \quad \textit{PartialCorr} = \begin{bmatrix} 1 & .6 & 0 \\ .6 & 1 & .6 \\ 0 & .6 & 1 \end{bmatrix}$$



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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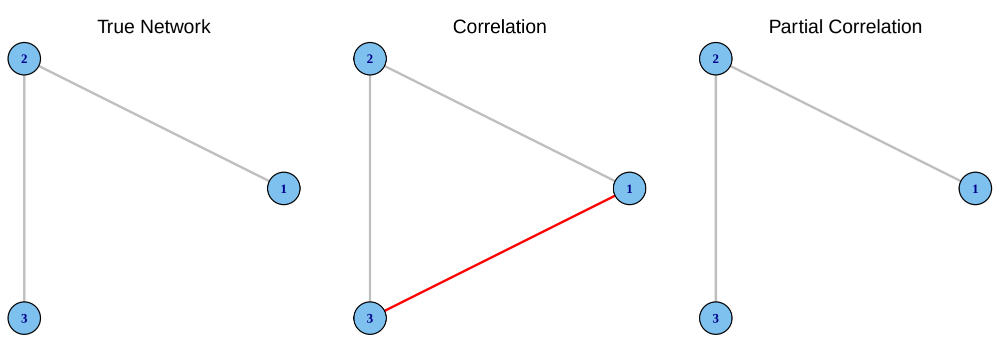


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## A Larger Example



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## A Larger Example

- A network with 10 nodes and 20 edges

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## A Larger Example

- A network with 10 nodes and 20 edges
- $n = 100$ observations



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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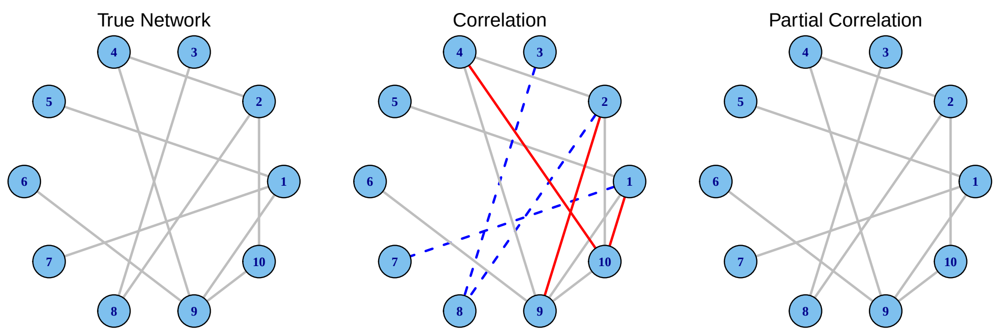






# Gaussian Graphical Models (GGMs)

## Partial Correlation for Gaussian Random Variables



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Partial Correlation for Gaussian Random Variables

- For Gaussian (multivariate normal) random variables, partial correlation between $X_i$ and $X_j$ given all other variables is given by the inverse of the (standardized) covariance matrix $\Sigma$.

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  - The $(i, j)$ entry in $\Sigma^{-1}$ gives the partial correlation between $X_i$ and $X_j$ given all other variables $X_{\setminus i,j}$.

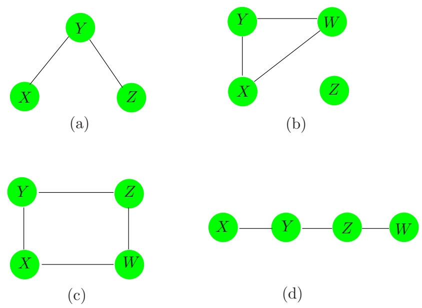


Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Partial Correlation for Gaussian Random Variables

- For Gaussian (multivariate normal) random variables, partial correlation between $X_i$ and $X_j$ given all other variables is given by the inverse of the (standardized) covariance matrix $\Sigma$.
  - The $(i, j)$ entry in $\Sigma^{-1}$ gives the partial correlation between $X_i$ and $X_j$ given all other variables $X_{\setminus i,j}$.
  - Multivariate normal: $X \sim N(0, \Sigma)$
  - $\Theta \equiv \Sigma^{-1}$ = inverse covariance/precision/concentration matrix.
  - Zeros in $\Theta \implies$ conditional independence!
  - Edges correspond to non-zeros in $\Theta$.

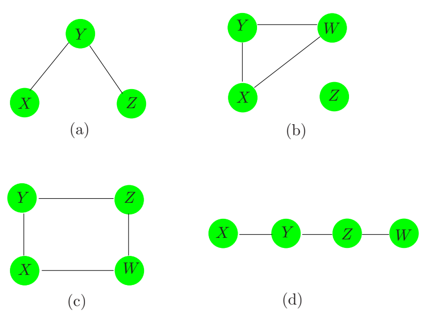
# Partial Correlation for Gaussian Random Variables






Introduction
Marginal Association Networks
**Conditional Independence Graphs**
**Gaussian Graphical Models**
Graphical Models for Other Distributions

# Partial Correlation for Gaussian Random Variables




$$\begin{pmatrix} - & \times & 0 \\ \times & - & \times \\ 0 & \times & - \end{pmatrix} \quad \begin{pmatrix} - & \times & \times & 0 \\ \times & - & \times & 0 \\ \times & \times & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix}$$

$$\begin{pmatrix} - & \times & 0 & \times \\ \times & - & \times & 0 \\ 0 & \times & - & \times \\ \times & 0 & \times & - \end{pmatrix} \quad \begin{pmatrix} - & 0 & 0 & \times \\ 0 & - & \times & 0 \\ 0 & \times & - & \times \\ \times & 0 & \times & - \end{pmatrix}$$

# Estimating GGMs

From the discussion so far, to estimate the network, we can

1. Calculate the empirical covariance matrix: for (centered) $n \times p$ data matrix $X$, $S = (n-1)^{-1}X^\top X$.
2. Get the inverse of $S$. Non-zero values of $S^{-1}$ give the edges.



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

# Estimating GGMs

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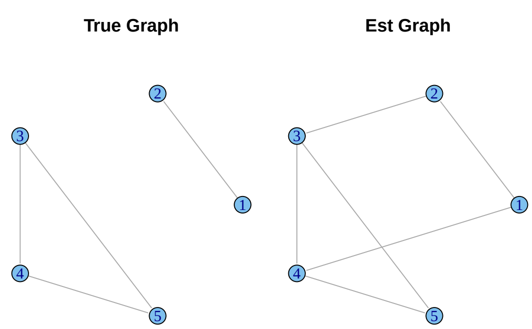
2. Get the inverse of $S$. Non-zero values of $S^{-1}$ give the edges.

While simple, this may not work well in practice, even with large samples!



## Estimating GGMs in High Dimensions

Many problems arise in high-dimensional settings, when $p \gg n$.



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Estimating GGMs in High Dimensions

Many problems arise in high-dimensional settings, when $p \gg n$.

- First, $S$ is not invertible if $p > n$!
- Even if $p < n$, but $n$ is not very large, we may still get poor estimates, and many false positives/negatives.

# Estimating GGMs in High Dimensions

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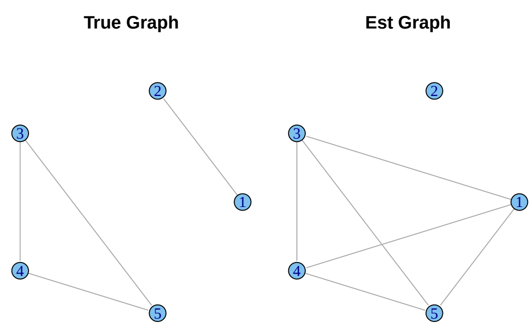






# Estimating GGMs in High Dimensions

- A number of methods have been recently proposed for estimating GGMs in high dimensions.
- The main idea in most of these methods is to use a regularization penalty, like the lasso.
- We discuss two approaches:
  - neighborhood selection
  - graphical lasso

# The Lasso

- The lasso involves finding $\boldsymbol{\beta}$ that minimizes

$$\left\|\mathbf{y} - \sum_{k=1}^{p} \mathbf{X}_k \boldsymbol{\beta}_k\right\|^2 + \lambda \sum_j |\beta_k|.$$



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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  - When $\lambda = 0$, we get least squares!
  - When $\lambda$ is very large, we get $\hat{\beta} = 0$.

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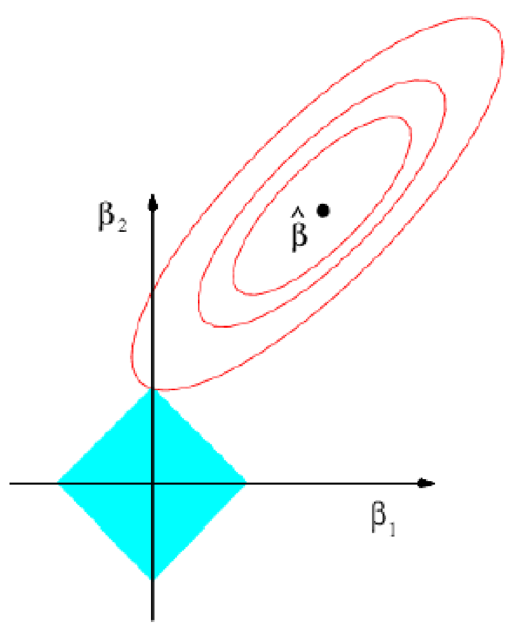
- Here $\lambda$ is a tuning parameter
  - When $\lambda = 0$, we get least squares!
  - When $\lambda$ is very large, we get $\hat{\beta} = 0$.
- Equivalently, find $\boldsymbol{\beta}$ that minimizes
$$\left\|\mathbf{y} - \sum_{k=1}^{p} \mathbf{X}_k \boldsymbol{\beta}_k\right\|^2$$
subject to the constraint that
$$\sum_{k=1}^{p} |\beta_k| \leq s.$$

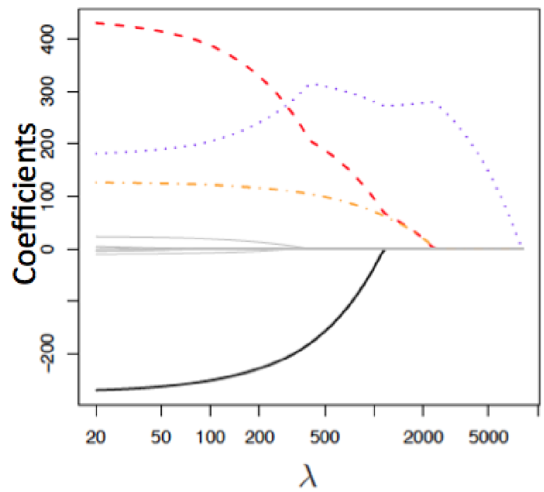




# A Geometric Interpretation



# Lasso As $\lambda$ Varies








# Estimating GGMs in High Dimensions – Method 1

The idea behind neighborhood selection, is to estimate the graph by fitting a penalized regression of each variable on all other variables.

- Find neighbors of each node $X_j$ by $l_1$-penalized regression or lasso:

$$\underset{\beta^j}{\text{minimize}} \quad \|X_j - X_{\neq j}\beta^j\|_2^2 + \lambda \sum_{k \neq j} |\beta_k^j|$$

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- The final estimate is found by combining all of the edges from these individual regression problems.
  - Symmetry — $\beta_k^j$ not always same as $\beta_j^k$.
  - Use min or max rule.





# Estimating GGMs in High Dimensions – Method 2

Estimate a sparse $\Theta$ via penalized maximum likelihood estimation (MLE).

Graphical Lasso (`glasso`)

$$\underset{\Theta}{\text{maximize}} \quad \log\det(\Theta) - \text{tr}(S\Theta) - \lambda\|\Theta\|_1$$

- Blue: Log-likelihood; $\log\det$ denotes the logarithm of the determinant of $\Theta$ and $\text{tr}$ the trace (sum of diagonal elements) $S\Theta$.
- Red: Penalty term encourages zeros on the off-diagonal elements of $\Theta$.

## Comparing the Two Approaches

- Neighborhood selection is an approximation for graphical lasso:
  - Consider regression of $X_j$ on $X_k, j \neq k$
  - Then, the regression coefficient for neighborhood selection is related to the $j, k$ element of $\Theta$:

$$\beta_k^j = -\frac{\Theta_{jk}}{\Theta_{jj}}$$



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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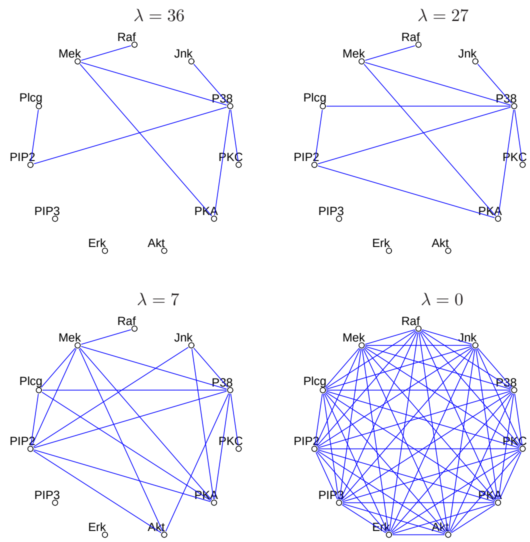
- Neighborhood selection is computationally more efficient, and may gives better estimates, but doesn't give an estimate of $\Theta$!

# A Real Example

- Flow cytometry proteomics in single cells (Sachs et al, 2003).
- $p = 11$ proteins measured in $n = 7466$ cells








# How to Choose $\lambda$?

- $\lambda$ modulates trade-off between model fit and network sparsity:
  - $\lambda = 0$ gives a dense network (no sparsity).
  - As $\lambda$ increases, network becomes more sparse.
- A number of approaches proposed in the literature and used in practice
  1. Cross-Validation — tends to yield overly dense networks.
  2. Extended BIC — adjusted BIC for high dimensions.
  3. Controlling the probability of falsely connecting disconnected components at level $\alpha$ (Banerjee et al, 2008):
     $$\lambda(\alpha) = \frac{t_{n-2}(\alpha/2p^2)}{\sqrt{n-2+t_{n-2}(\alpha/2p^2)}},$$
     ($t_{n-2}(\alpha)$ is the $(100-\alpha)\%$ quantile of $t$-dist with $n-2$ d.f.)
  4. Stability selection — Choose $\lambda$ that gives the most **stable network** (R-package `huge`)

# Other Types of Graphical Models





## Nonparanormal (Gaussian Copula) Models

- Suppose $X \nsim N(0, \Sigma)$, but there exist monotone functions $f_j, j = 1, \ldots p$ such that $[f_1(X_1), \ldots f_p(X_p)] \sim N(0, \Sigma)$

# Nonparanormal (Gaussian Copula) Models

- Suppose $X \not\sim N(0, \Sigma)$, but there exist monotone functions $f_j, j = 1, \ldots p$ such that $[f_1(X_1), \ldots f_p(X_p)] \sim N(0, \Sigma)$
  - $X$ has a nonparanormal distribution $X \sim NPN_p(f, \Sigma)$.
  - $f$ and $\Sigma$ are parameters of the distribution, and estimated from data.
  - For continuous distributions, the nonparanormal family is the same as the Gaussian copula family



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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- To estimate the nonparanomal network:

  i) transform the data: $[f_1(X_1), \ldots f_p(X_p)]$
  ii) estimate the network of the transformed data (e.g. calculate the empirical covariance matrix of the transformed data, and apply glasso or neighborhood selection)

## A Related Procedure

- Liu et al (2012) and Xue & Zou (2012) proposed a closely related idea using rank-based correlation
  - Let $r_j^i$ be the rank of $x_j^i$ among $x_j^1, \ldots, x_j^n$ and $\bar{r}_j = (n+1)/2$ be the average rank
  - Calculate Spearman's $\rho$ or Kendall's $\tau$

$$\hat{\rho}_{jk} = \frac{\sum_{i=1}^n (r_j^i - \bar{r}_j)(r_k^i - \bar{r}_k)}{\sqrt{\sum_{i=1}^n (r_j^i - \bar{r}_j)^2 \sum_{i=1}^n (r_k^i - \bar{r}_k)^2}}$$

$$\hat{\tau}_{jk} = \frac{2}{n(n-1)} \sum_{1 \le i < i' \le n} \mathrm{sign}\left( (x_j^i - x_j^{i'})(x_k^i - x_k^{i'}) \right)$$





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- If $X \sim NPN_p(f, \Sigma)$, then $\Sigma_{jk} = 2\sin(\rho_{jk}\pi/6) = \sin(\tau_{jk}\pi/2)$
- Therefore, we can estimate $\Sigma^{-1}$ by plugging in rank-based correlations into graphical lasso (R-package `huge`)

# A Real Data Example

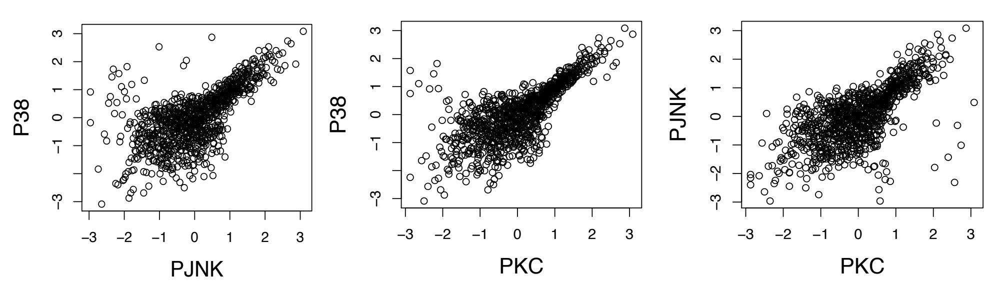
- Protein cytometry data for cell signaling (Sachs et al, 2005)
- Transform the data using a Gaussian copula (Liu et al, 2009), giving marginal normality



Introduction
Marginal Association Networks
**Conditional Independence Graphs**
**Gaussian Graphical Models**
Graphical Models for Other Distributions

# A Real Data Example

- Protein cytometry data for cell signaling (Sachs et al, 2005)
- Transform the data using a Gaussian copula (Liu et al, 2009), giving marginal normality
- Pairwise relationships still seem non-linear




- Shapiro-Wilk test rejects multivariate normality: $p < 2 \times 10^{-16}$

## Graphical Models for Discrete Random Variables

- In many cases, biological data are not Gaussian: SNPs, RNAseq, etc



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Graphical Models for Discrete Random Variables

- In many cases, biological data are not Gaussian: SNPs, RNAseq, etc
- Need to estimate CIG for other distributions: binomial, poisson, etc

# Graphical Models for Discrete Random Variables

- In many cases, biological data are not Gaussian: SNPs, RNAseq, etc
- Need to estimate CIG for other distributions: binomial, poisson, etc
- In this case, the estimators do not have a closed-form!
- A special case, which is computationally more tractable, is the class of pairwise MRFs





# Pairwise Markov Random Fields

[3] Wainwright & Jordan (2008)

## Pairwise Markov Random Fields

- The idea of pairwise MRFs is to "assume" that only two-way interactions among variables exist
  - The pairwise MRF associated with graph $G$ over the random vector $X$ is the family of probability distributions $P(X)$ that can be written as

$$P(X) \propto \exp \sum_{(j,k) \in E} \phi_{jk}(x_j, x_k)$$

  - For each edge $(j, k) \in E$, $\phi_{jk}$ is called the edge potential function
- For discrete random variables, any MRF can be transformed to an MRF with pairwise interactions by introducing additional variables[3]

[3] Wainwright & Jordan (2008)



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Graphical Models for Binary Random Variables

## Graphical Models for Binary Random Variables

- Suppose $X_1, \dots, X_p$ are binary random variables, corresponding to, e.g. SNPs, or DNA methylation



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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- Suppose $X_1, \dots, X_p$ are binary random variables, corresponding to, e.g. SNPs, or DNA methylation
- A special case of discrete graphical models is the Ising model for binary random variables

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$$P_\theta(x) = \frac{1}{Z(\theta)} \exp\left\{ \sum_{(j,k)\in E} \theta_{jk} x_j x_k \right\}$$

  - A pairwise MRF for binary data, with $\phi_{jk}(x_j, x_k) = \theta_{jk} x_j x_k$
  - $x^i \in \{-1, +1\}^p$
  - The partition function $Z(\theta)$ ensures that distribution sums to 1
  - $(j, k) \in E$ iff $\theta_{jk} \neq 0$!



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## Graphical Models for Binary Random Variables

[4] Ravikumar et al (2010)

## Graphical Models for Binary Random Variables

- We can consider a neighborhood selection[4] approach with an $\ell_1$ (lasso) penalty to find the neighborhood of each node $N(j) = \{k \in V : (j,k) \in E\}$

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Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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- For $j = 1, \ldots, p$, need to solve (after some algebra)

$$\min_\theta \left\{ n^{-1} \sum_{i=1}^{n} \left[ f(\theta; x^i) - \sum_{k \neq j} \theta_{jk} x_j^i x_k^i + \lambda \|\theta_{-j}\|_1 \right] \right\}$$

  - $f(\theta; x) = \log\left\{\exp\left(\sum_{k\neq j} \theta_{jk} x_k\right) + \exp\left(-\sum_{k\in -j} \theta_{jk} x_k\right)\right\}$

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  - $f(\theta; x) = \log\left\{\exp\left(\sum_{k\neq j} \theta_{jk} x_k\right) + \exp\left(-\sum_{k \in -j} \theta_{jk} x_k\right)\right\}$
- This is equivalent to solving $p$ penalized logistic regression problems, which is straightforward (R-package `glmnet`)

[4] Ravikumar et al (2010)





# Other Non-Gaussian Distributions

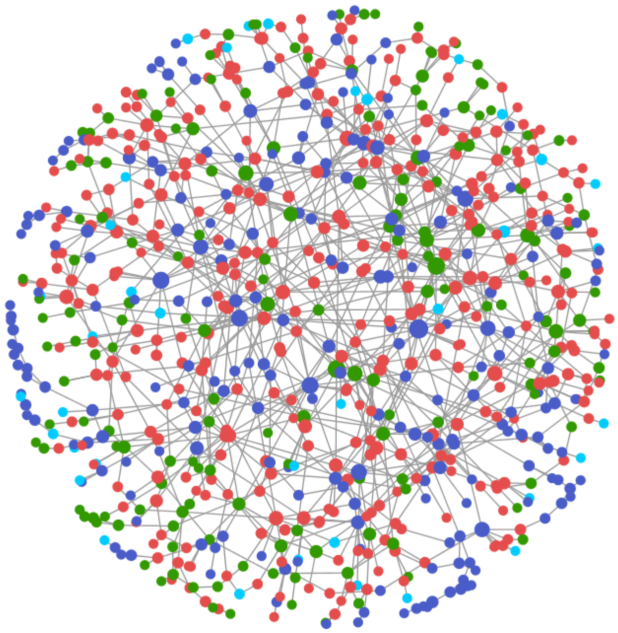
- Assume a pairwise graphical model

$$P(X) \propto \exp\left\{ \sum_{j \in V} \theta_j \phi_j(X_j) + \sum_{(j,k) \in E} \theta_{jk} \phi_{jk}(X_j, X_k) \right\}$$

- Then, similar to the Ising model, graphical models can be learned for other members of the exponential family
  - Poisson graphical models (for e.g. RNAseq), Multinomial graphical models, etc
  - All of these can be learned using a neighborhood selection approach, using the `glmnet` package[5]
  - We can even learn networks with multiple types of nodes (gene expression, SNPs, and CNVs)[6]

[5] Yang et al (2012)
[6] Yang et al (2014), Chen et al (2015)

# Mixed Graphical Models







# A General Approach for Estimation of Graphical Models

- Consider $n$ iid observations from a $p$-dimensional random vector $x = (X_1, \ldots, X_p) \sim \mathcal{P}$
- Consider the (undirected) graph $G = (V, E)$ with vertices $V = \{1, \ldots, p\}$
- Want to estimate edges $E \subset V \times V$ that satisfy $\forall j \in V$, $\exists N(j)$ such that:

$$p_j(X_j | \{X_k, k \neq j\}) = p_j(X_j | \{X_k : k \in N(j)\}) = p_j(X_j | \{X_k : (k, j) \in E\})$$

- $N(j)$ is the minimal set of variables on which the conditional densities depend

## Estimating Conditional Independencies

Question: how to condition?



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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Question: how to condition?

- Approach 1: Estimate the joint density $f(X_1, \ldots, X_p)$; then get the conditionals $f_j(X_j \mid X_{-j})$

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## Estimating Conditional Independencies

Question: how to condition?

- Approach 1: Estimate the joint density $f(X_1, \ldots, X_p)$; then get the conditionals $f_j(X_j \mid X_{-j})$
  - Efficient, coherent
  - Computationally challenging
  - Restrictive: how many joint distributions do you know?
  - Hard to check if assumptions hold!



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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- Approach 2: Estimate the conditionals directly $f_j(X_j \mid X_{-j})$
  - Computationally easy
  - Leads to easy & flexible models (regression)!
  - May not be efficient or coherent



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

## A Semi-parametric Approach

## A Semi-parametric Approach

- Consider additive non-linear relationships (additive model):

$$X_j \mid X_{-j} = \sum_{k \neq j} f_{jk}(X_k) + \varepsilon$$



Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

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- In other words, we assume that conditional distributions and conditional means depend on the same set of variables
- We then use a semi-parametric approach for estimating the conditional dependencies

# SpaCE JAM[7]

- Sparse Conditional Estimation with Jointly Additive Models (SpaCE JAM)

$$\underset{f_{jk}\in\mathcal{F}}{\text{minimize}} \frac{1}{2n}\sum_{j=1}^{p}\left\|x_j - \sum_{k\neq j} f_{jk}(x_k)\right\|_2^2 + \lambda \sum_{k>j}\left(\|f_{jk}(x_k)\|_2^2 + \|f_{kj}(x_j)\|_2^2\right)^{1/2}$$

  - $f_{jk}(x_k) = \Psi_{jk}\beta_{jk}$
  - $\Psi_{jk}$ is a $n \times r$ matrix of basis functions for $f_{jk}$
  - $\beta_{jk}$ is an $r$-vector of coefficients
  - The standardized group lasso penalty for functions $\|f_{jk}\|_2$
- This is a convex problem, and block coordinate descent converges to the global minimum

[7] Voorman et al (2014), R-package `spacejam`



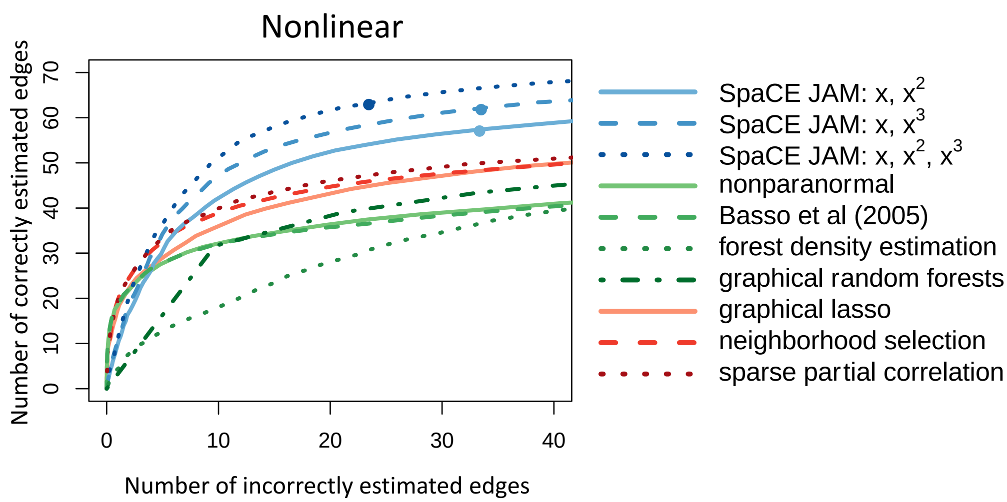


# Other Flexible Procedures

- Forest density estimation (Liu et al, 2011) assumes that underlying graph is a forest, and estimates the bivariate densities non-parametrically.
- Graphical random forests (Fellinghauer et al, 2013) uses random forests to flexibly model conditional means
  - They consider conditional dependencies through conditional mean
  - They allow for general random variables, discrete or continuous
  - Use a random forest to estimate $E[X_j \mid X_{\setminus j}]$ non-parametrically
  - Theoretical properties have not yet been justified

# Comparison on Simulated Data

non-linear relationships ($p = 100,\ n = 50$)

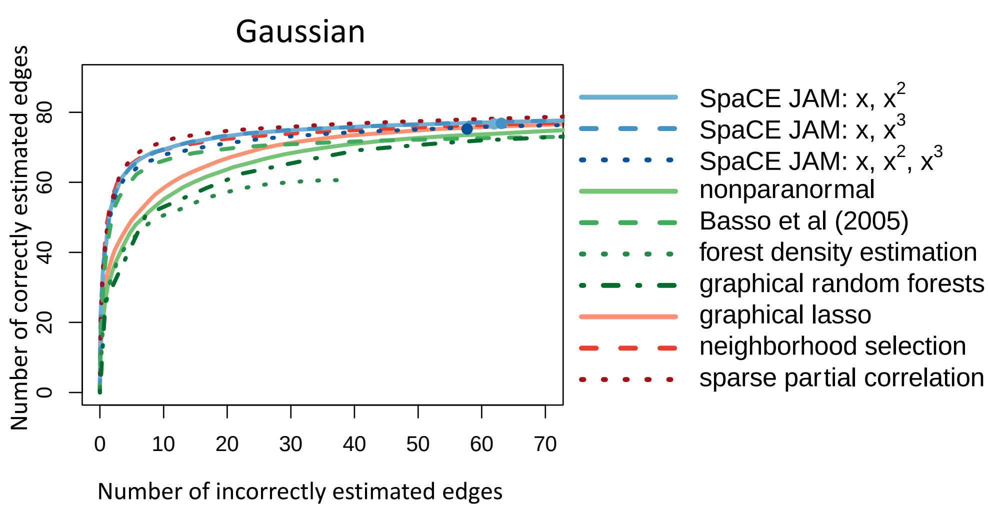




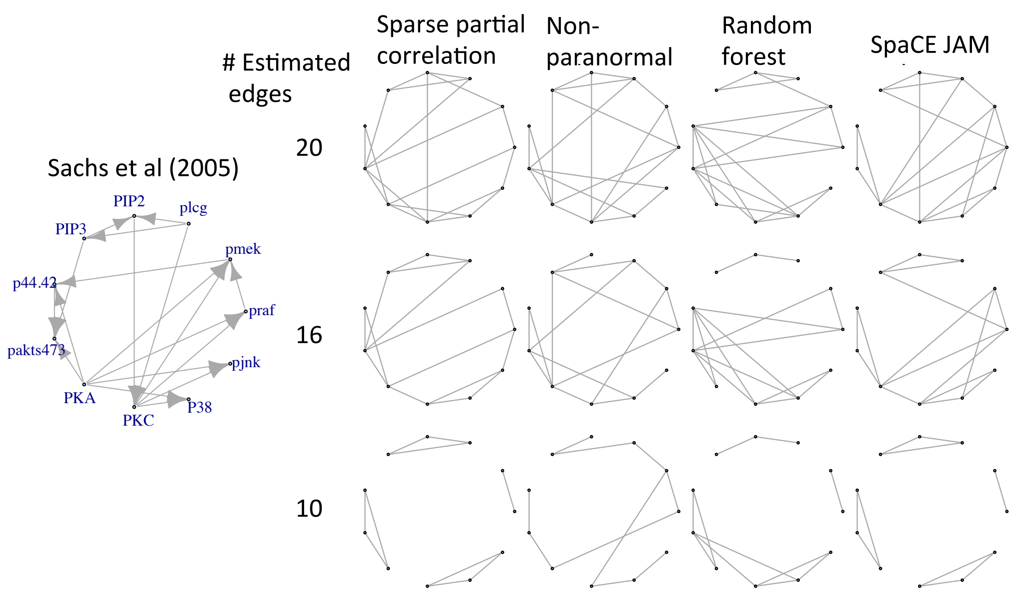

Introduction
Marginal Association Networks
Conditional Independence Graphs
Gaussian Graphical Models
Graphical Models for Other Distributions

# Comparison on Simulated Data

linear relationships ($p = 100,\ n = 50$)



# Estimation of Cell Signaling Network








# Other Extensions of GGMs

- Multiple Graphical Models
  - For groups of observations, estimate graphical models with shared structure across groups and individual structure within groups.
- Time Varying Graphical Models
  - Smoothly varying graph over time estimated via local kernel smoothers.
  - Change points in graph structure over time estimated via fusion penalties.
- Latent Variable Graphical Models
  - Assume observed features are dependent on latent variables which exhibit a low-rank effect. Estimate a sparse (graph structure) plus low-rank inverse covariance matrix.

# Pathway & Network Analysis of Omics Data: Learning Directed Networks

Ali Shojaie
Department of Biostatistics
University of Washington
faculty.washington.edu/ashojaie

Summer Institute for Statistical Genetics – 2023





## Bayesian Networks

## Bayesian Networks

- Bayesian networks are a special class of graphical models defined on directed acyclic graphs.



Introduction
Estimating DAGs
DAGs for Time Series Data

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- Directed acyclic graphs (DAGs) are defined as graphs that:
  i) only have directed edges, i.e. if $A_{ij} \neq 0$, $A_{ji} = 0$;
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Introduction
Estimating DAGs
DAGs for Time Series Data

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- Bayesian networks are widely used to model causal relationships between variables.
- Note that correlation $\neq$ causation!
- Therefore, we (usually) cannot estimate Bayesian networks from (partial) correlations





## Why Bayesian Networks?

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Many biological networks include directed edges:





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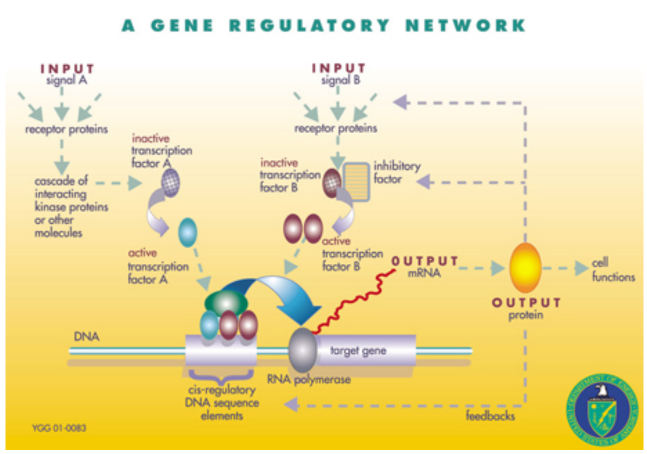
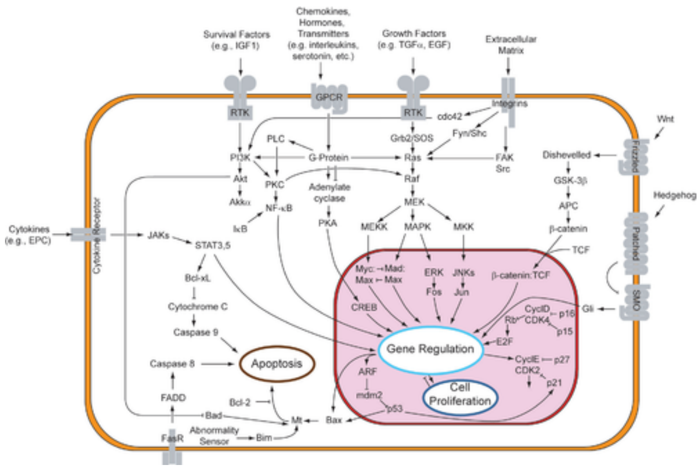
- In gene regulatory networks, protein products of transcription factors can alter the expression of target genes, but the target genes (usually) don't have a direct effect on the expression of transcription factors



## Why Bayesian Networks?

Many biological networks include directed edges:

- In cell signaling networks, the signal from the cell's environment is transducted into the cell, and results e.g. in (global) changes in gene expression, but gene expression may not affect the environmental factors



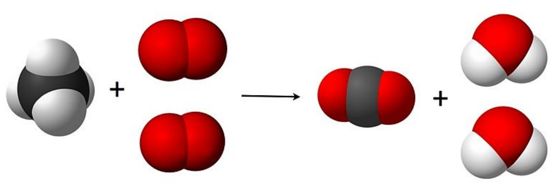





## Why Bayesian Networks?

Many biological networks include directed edges:

- Biochemical reactions in metabolic networks, may not reversible, and in that case, one metabolite may affect the other, but the relationship is ont reciprocated



## Why Bayesian Networks?

However, biological networks may not be DAGs:

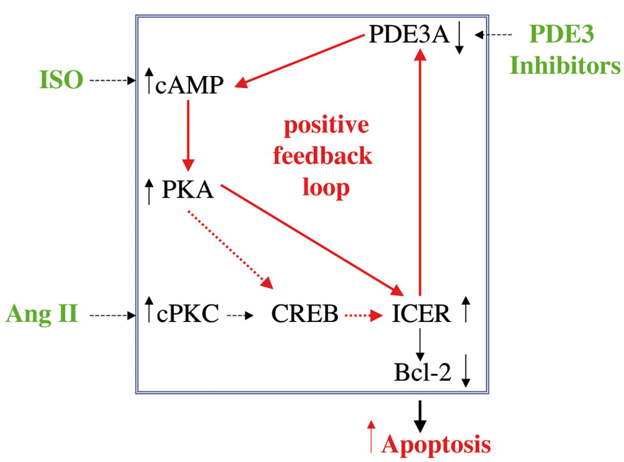


Introduction
Estimating DAGs
DAGs for Time Series Data

## Why Bayesian Networks?

However, biological networks may not be DAGs:

- Gene regulatory networks, signaling networks and metabolic networks, may all contain feedback loops (positive/negative)




which make estimation even more difficult!

## What's the Difference?



Introduction
Estimating DAGs
DAGs for Time Series Data

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- Undirected networks (e.g. GGM) provide information about associations among variables; while this greatly helps in the study of biological systems, in some cases, they are not enough (e.g. drug development).



Introduction
Estimating DAGs
DAGs for Time Series Data

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## What's the Difference?

- Bayesian networks are widely used to model causal relationships between variables.
- Undirected networks (e.g. GGM) provide information about associations among variables; while this greatly helps in the study of biological systems, in some cases, they are not enough (e.g. drug development).
- The main difference is the direction of the edges; however, it turns out that there are also some differences in terms of structure/skeleton of the network (more on this later).
- We can estimate undirected networks from observational data, i.e. steady-state gene expression data, but usually they are not enough for estimation of directed networks
- Finally, estimating directed networks is (much) more difficult





## Why is estimation more difficult?

- Estimation of Bayesian networks requires estimating both the skeleton of the network (i.e. whether there is an edge between $i$ and $j$) and also the direction of the edges.

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- Then, no matter what $n$ is, we cannot distinguish between $X_1 \to X_2$ and $X_2 \to X_1$, so basically what we see is:

# Directed Graphs: Some Terminology

- The parents of node $j$ are $\{k : k \to j\}$, we denote this by $\mathrm{pa}_j$ or $\mathrm{pa}(j)$
- The children of node $j$ are $\{k : j \to k\}$
- Two vertices connected by an edge are called adjacent



Introduction
Estimating DAGs
DAGs for Time Series Data

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- A path between two nodes $i$ and $j$ is a sequence of distinct adjacent nodes:
  - e.g. $i \leftarrow k_1 \to k_2 \to k_3 \leftarrow j$
  - In a DAG with $p$ nodes, there cannot be a path longer than $p-1$ (why?)
  - There can be multiple paths between two nodes

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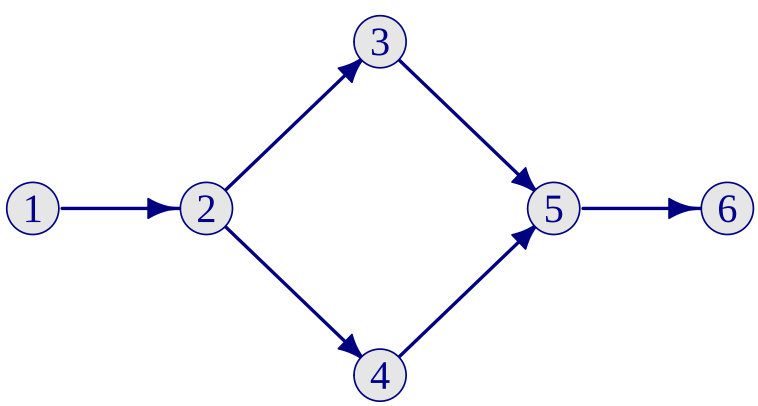
  - In a DAG with $p$ nodes, there cannot be a path longer than $p - 1$ (why?)
  - There can be multiple paths between two nodes
- $i$ is an ancestor of $j$ if there is a directed path of length $\geq 1$ from $i$ to $j$: $i \to \cdots \to j$ (or if $i = j$)
- If $i$ is an ancestor of $j$, then $j$ is said to be a descendant of $i$

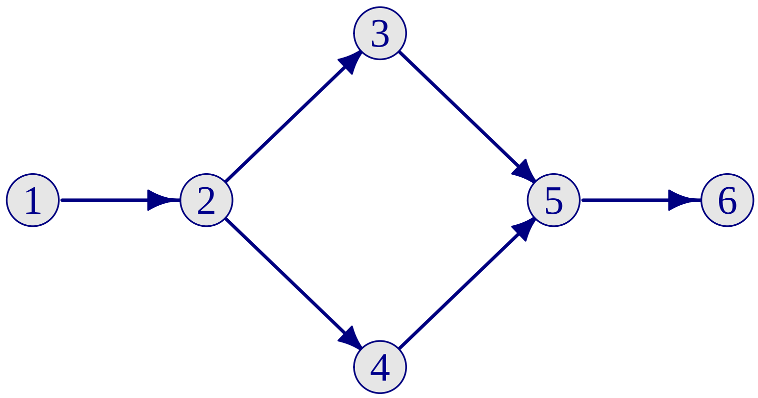




## Directed Graphs: Some Terminology



## Directed Graphs: Some Terminology




- What are parents/children of $\{1, \ldots 5\}$?
- What are paths between 1&4, 3&4, 2&6?
- What are ancestors of $\{1, \ldots 5\}$?





## Directed Graphs: Some Terminology

An important concept in DAGs is colliders (aka "inverted forks"):

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- $k$ is a collider on a path between $i$ and $j$ if it is a not an end-point of the path, and the path is of the form

$$i \ldots \rightarrow k \leftarrow \ldots j$$



Introduction
Estimating DAGs
DAGs for Time Series Data

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  - $i \ldots \leftarrow k \leftarrow \ldots j$
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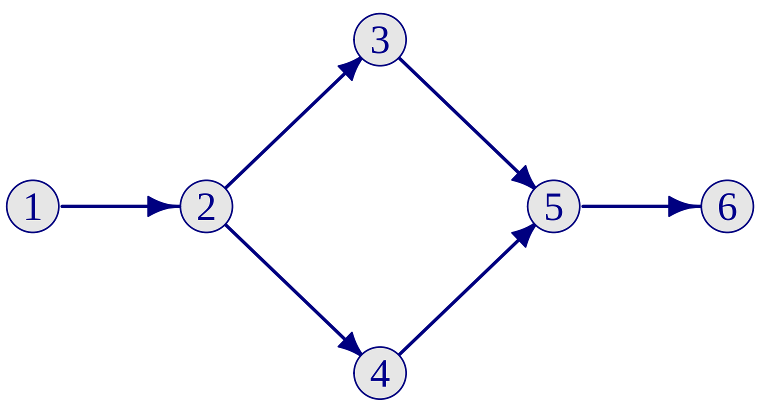
$$i \ldots \to k \leftarrow \ldots j$$

- $k$ is an non-collider if it is not an end-point, and is not a collider on a path:
  - $i \ldots \leftarrow k \leftarrow \ldots j$
  - $i \ldots \to k \to \ldots j$
  - $i \ldots \leftarrow k \to \ldots j$
- <u>Note</u>: colliders and non-colliders are defined w.r.t. paths; a collider in one path can be a non-collider in another!





## Directed Graphs: Some Terminology



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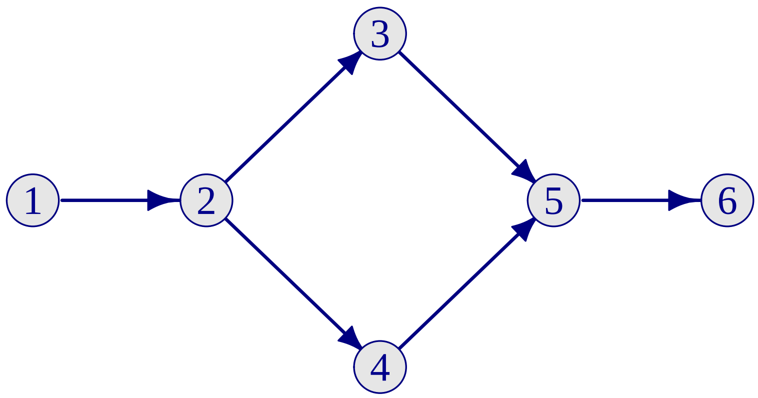


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Introduction
Estimating DAGs
DAGs for Time Series Data

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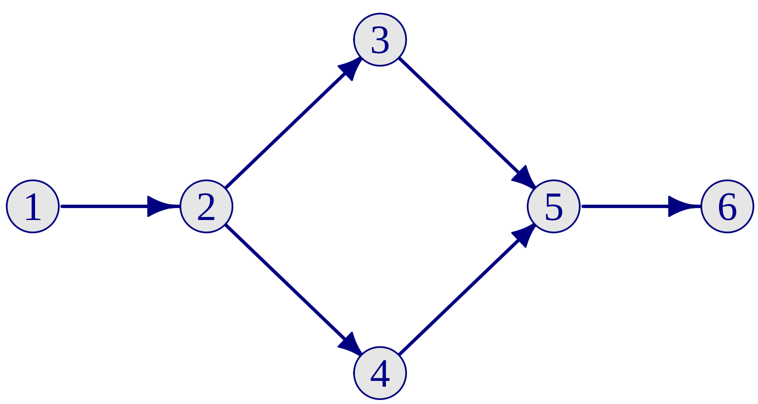
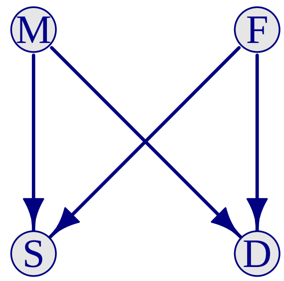



- What are the colliders on paths between 1&4, 3&4, 2&6?
- What are the non-colliders on paths between 1&4, 3&4, 2&6?

## Estimating Directed Graphs

- The presence of colliders makes the estimation of directed graphs very challenging...



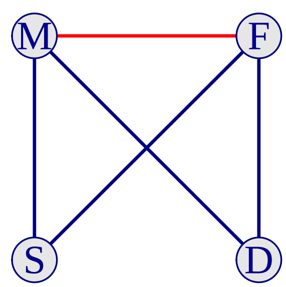

  - Genetic information for *M*other, *F*ather, *D*aughter and *S*on in form of dominant/recessive genotype (A/a) for a single gene
  - Then each individual can have one of three states: AA, aa, Aa





## Estimating Directed Graphs

- Conditioning on all other nodes, gives additional moral (!!) edges ($\Rightarrow$ moral graph)



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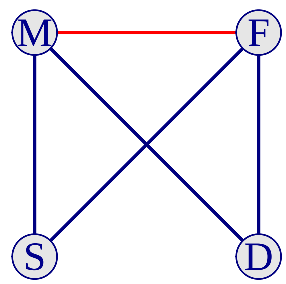



- Learning the skeleton of DAGs from observational data requires finding right conditioning set
  - Naively, this is done by *searching over all possible subset of other $p-2$ nodes* — NP-hard with complexity $O(2^{p^2})$!!





# Estimation of DAGs from Observational Data

Two general classes of algorithms for estimating DAGs:

- constraint-based methods
  - Often based on tests for CI; provide theoretical guarantees
  - PC algorithm, Grow-Shrink
- score & search methods
  - They assign a "score" to each estimated graph (e.g. based on likelihood, Bayes factor, AIC etc)
  - Greedy search to find the best scoring graph (Hill Climbing)
- "hybrid" methods
  - Usually first find the Markov blanket (e.g. the moral graph)
  - Then search in a restricted space (Max-Min Hill Climbing)

# Constraint-Based Methods

- Need a conditional independence test (to test if $X \perp\!\!\!\perp Y \mid Z$)
  - For Gaussian data, we can use partial correlation (or the Fisher's Z-transformation of it)
  - For Binary data, we can use logOR
  - In general, we can use conditional mutual information
- The idea is to see if there exists a set $S$, for each pair of nodes $j, j'$, such that $X_j \perp\!\!\!\perp X_{j'} \mid S$
  - $S$ can have 0 to p-2 members! usually stop at some $k \ll p$
  - I.e., for each pair of variables (all $\binom{p}{2}$ of them), we need to look at all possible subsets of remaining variables!!
- These methods find the DAG skeleton (*conditional independence is symmetric*) — will talk about direction later





# PC Algorithm (Spirtes et al, 1993)

- One of the first algorithms for learning structure of DAGs
- Efficient implementations that allow for learning DAG structures with $p$ up to $\sim 1000$
  - R-package `pcalg` (Kalisch & Buhlmann, 2007)
- The algorithm starts with a complete graph (i.e. fully connected)
- Then for each pair of nodes $j, j'$ it finds a separating set, $S$ such that $X_j \perp\!\!\!\perp X_{j'} \mid S$
- If a set is found, then remove the edge, otherwise, $j - j'$

# PC Algorithm (Spirtes et al, 1993)

Start with a complete undirected graph, and set $i = 0$

Repeat

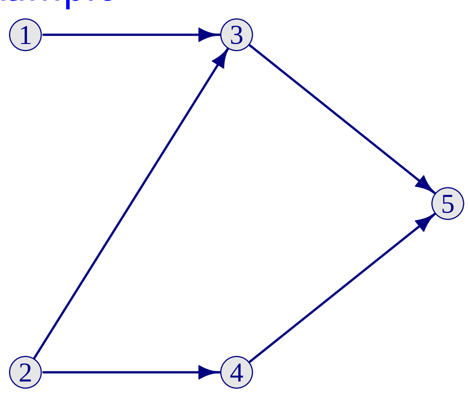
- For each $j \in V$
- For each $j' \in \mathrm{ne}(j)$
- Determine if $\exists S \subset \mathrm{ne}(j) \backslash \{j'\}$ with $|S| = i$
  - Test for CI: is $X_j \perp\!\!\!\perp X_{j'} \mid S$?
  - If such an $S$ exists, then set $S_{jj'} = S$, remove $j - j'$ edge
- $i = i + 1$

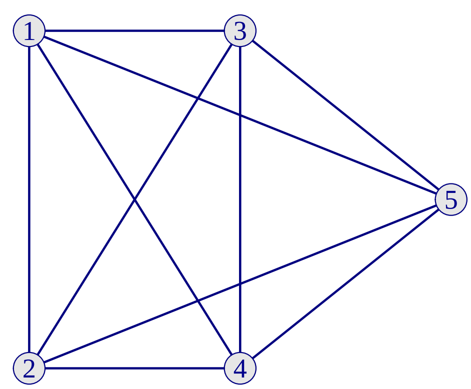
Until $|\mathrm{ne}(j)| < i$ for all $j$





# Example



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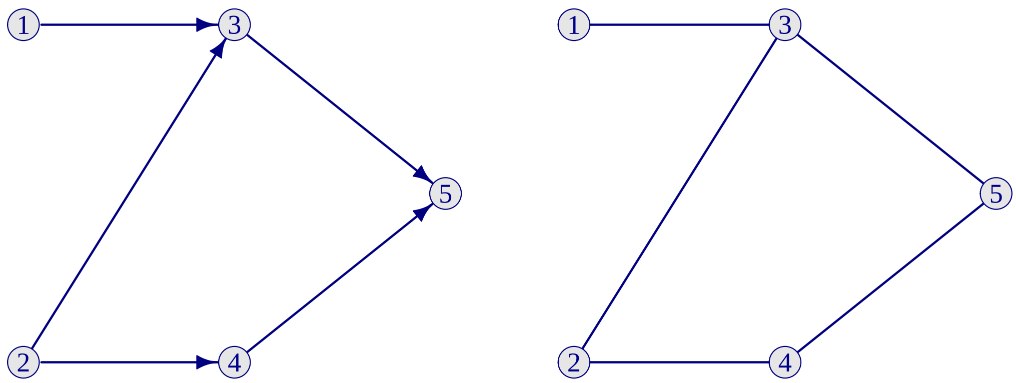



$$
\begin{array}{ll}
i = 0 & S_{1,2} = \emptyset \\
 & S_{1,4} = \emptyset \\
i = 1 & S_{3,4} = \{2\} \\
i = 2 & S_{1,5} = \{3,4\} \\
 & S_{2,5} = \{3,4\} \\
i = 3 & \text{STOP } (|\mathrm{ne}_j| < 3 \;\; \forall j)
\end{array}
$$



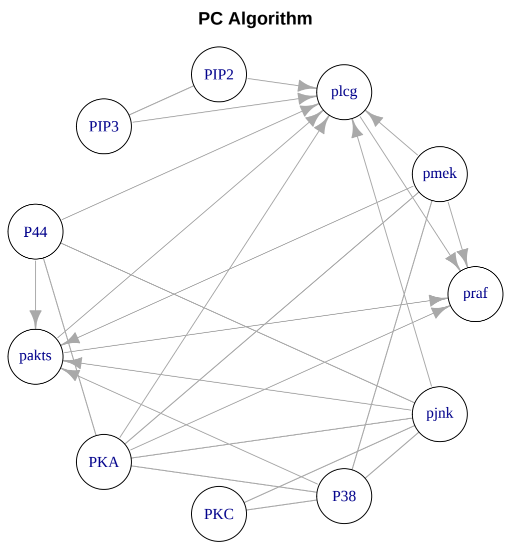


## Analysis of Protein Flow Cytometry using pcalg

```
> dat <- read.table('sachs.data')
> p <- ncol(dat)
> n <- nrow(dat)
## define independence test (partial correlations)
> indepTest <- gaussCItest
## define sufficient statistics
> suffStat <- list(C=cor(dat), n=n)
## estimate CPDAG
> pc.fit <- pc(suffStat, indepTest, p, alpha=0.1, verbose=FALSE)
> plot(pc.fit, main='PC Algorithm')
```

- Need to determine the type of CI test (`indepTest`), and sufficient statistics (`suffStat`)
- Also need to choose $\alpha$ (`alpha`), the probability of false positive for selecting edges.
  - Larger values of $\alpha$ allow more edges (not adjusted for multiple comparisons)
  - The algorithm works faster when $\alpha$ is small

## Analysis of Protein Flow Cytometry using `pcalg`



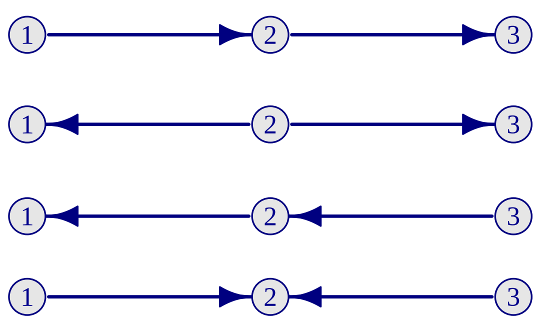

But wait, where did the directions come from? And why are only some of the edges directed?





## Markov Equivalence

Consider the following 4 graphs



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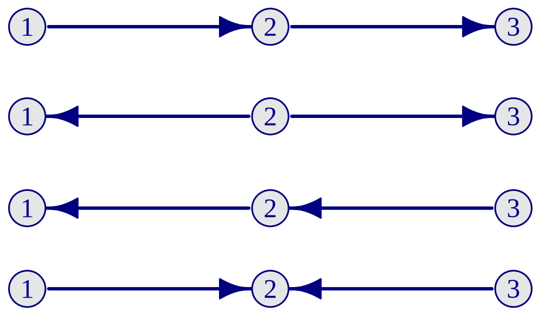

Which graphs satisfy $X_1 \perp\!\!\!\perp X_3 \mid X_2$?





# Markov Equivalence

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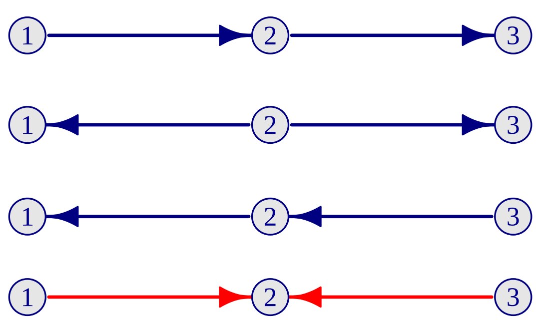

## Markov Equivalence

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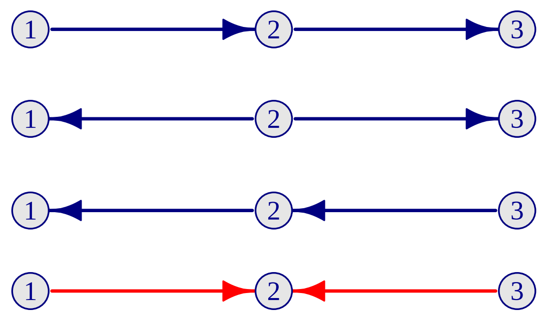



In the first 3 graphs, $X_1 \perp\!\!\!\perp X_3 \mid X_2$?
Two graphs that imply the same CI relationships via d-separation are called Markov equivalent

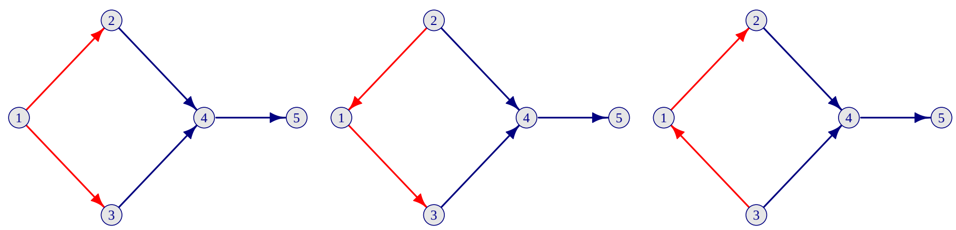




## Representation of Markov Equivalence

- Markov equivalent graphs correspond to the same probability distribution and cannot be distinguished from each other based on observations!
- Therefore, the direction of edges that correspond to Markov equivalent graphs cannot be determined
- We show these edges using undirected edges in the graph
- The resulting graph is a CPDAG (completed partially directed acyclic graph), and is really the best we can do!

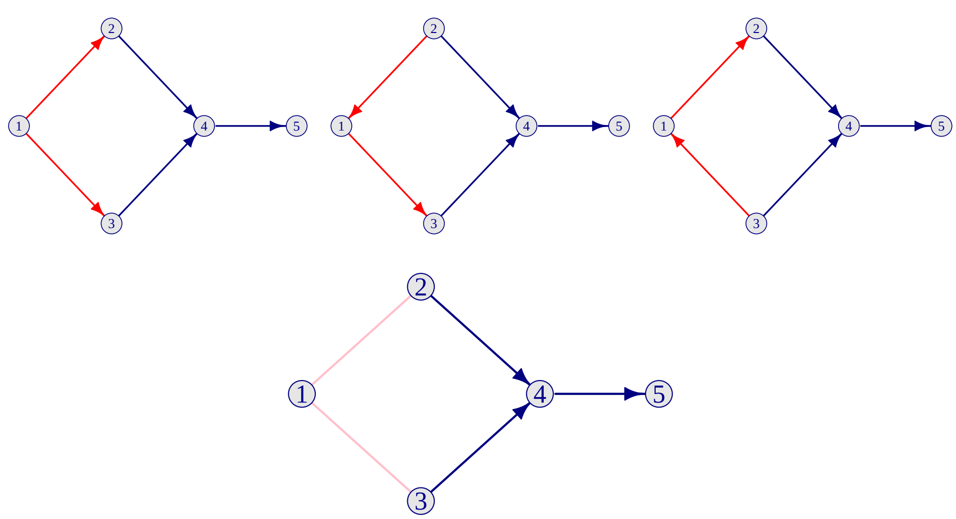
# CPDAGs






Introduction
Estimating DAGs
DAGs for Time Series Data
PC Algorithm
Other Estimation Methods

CPDAGs

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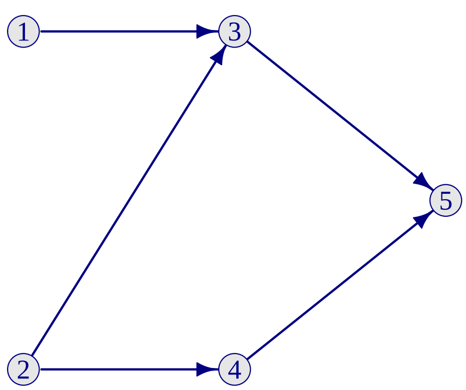
# Finding Partial Directions in DAGs

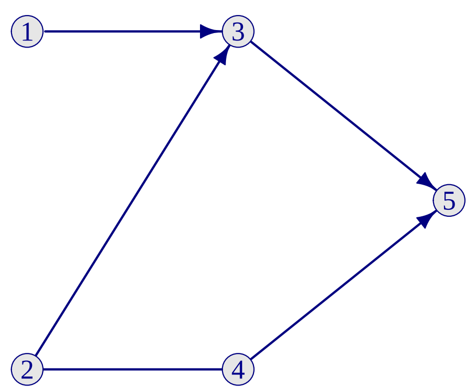
- Partial directions are determined from unmarried colliders:
  - For each unmarried collider $i - k - j$
  - If $k \notin S_{ij}$, orient $i - k - j$ as $i \rightarrow k \leftarrow j$
- In addition to the above rule,
  - Orient each remaining unmarried collider $i \rightarrow k - j$ as $i \rightarrow k \rightarrow j$
  - If $i \rightarrow k \rightarrow j$ and $i - j$ then orient as $i \rightarrow j$
  - If $i - m - j$ and $i \rightarrow k \leftarrow j$ are unmarried colliders and $m - k$, then orient as $m \rightarrow k$





# Example




$i = 0$ $\quad S_{1,2} = \emptyset$

$\quad S_{1,4} = \emptyset$

$i = 1$ $\quad S_{3,4} = \{2\}$

$i = 2$ $\quad S_{1,5} = \{3, 4\}$

$\quad S_{2,5} = \{3, 4\}$

# The bnlearn package

- There are a couple of R-packages for learning (CP)DAGs, including pclag, bnlearn, deal
- bnlearn implements a number of estimation methods, both constraint-based and search-based:
  - constraint-based algorithms:
    - Grow-Shrink (GS)
    - Incremental Association Markov Blanket (IAMB)
    - Fast Incremental Association (Fast-IAMB)
    - Interleaved Incremental Association (Inter-IAMB)
  - score-based algorithms:
    - Hill Climbing (HC)
    - Tabu Search (Tabu)
  - hybrid learning algorithms:
    - Max-Min Hill Climbing (MMHC)
    - General 2-Phase Restricted Maximization (RSMAX2)





# Analysis of Protein Flow Cytometry using bnlearn

```
> dag1 <- gs(dat, alpha=0.01)   #GS method
> dag2 <- hc(dat2)              #Hill-Climbing search
>
> par(mfrow= c(1,2))
> plot(dag1)
> plot(dag2)
>
> compare(dag1, dag2)           #compare the two DAGs
```

- For GS need to choose $\alpha$ (alpha), the false positive probability for selecting edges
- gs (and other structure-based methods) find a PCDAG
- hc gives a directed graph (with highest score)
  - Multiple criteria for choosing the "best" graph
  - To "search" the space either a new edge is added, or a current edge is removed, or reversed (if no cycles)

# Analysis of Protein Flow Cytometry using bnlearn

```
> dag1
  Bayesian network learned via Constraint-based methods

  model:
    [partially directed graph]
  nodes:                                 11
  arcs:                                  26
    undirected arcs:                     3
    directed arcs:                       23
  average markov blanket size:           6.00
  average neighbourhood size:            4.73
  average branching factor:              2.09

  learning algorithm:                    Grow-Shrink
  conditional independence test:         Pearson's Linear Correlation
  alpha threshold:                       0.01
  tests used in the learning procedure:  2029
  optimized:                             TRUE
```



Introduction
Estimating DAGs
DAGs for Time Series Data
PC Algorithm
Other Estimation Methods

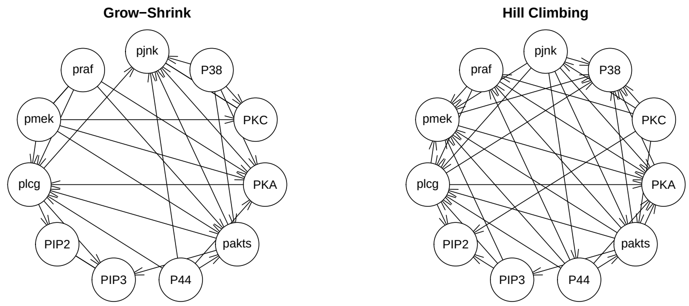
# Analysis of Protein Flow Cytometry using bnlearn

```
> dag2
  Bayesian network learned via Score-based methods

  model:
   [PKC][pjnk|PKC][P44|pjnk][pakts|P44:PKC:pjnk][praf|P44:pakts:PKC][PIP3|pakts
   [plcg|praf:PIP3:P44:pakts:pjnk][pmek|praf:plcg:PIP3:P44:pakts:pjnk]
   [PIP2|plcg:PIP3:PKC][PKA|praf:pmek:plcg:P44:pakts:pjnk]
   [P38|pmek:plcg:pakts:PKA:PKC:pjnk]
  nodes:                                 11
  arcs:                                  35
    undirected arcs:                     0
    directed arcs:                       35
  average markov blanket size:           8.00
  average neighbourhood size:            6.36
  average branching factor:              3.18

  learning algorithm:                    Hill-Climbing
  score:
                                        Bayesian Information Criterion (Gaussia
  penalization coefficient:              4.459057
  tests used in the learning procedure:  505
  optimized:                             TRUE
```

## Analysis of Protein Flow Cytometry using `bnlearn`



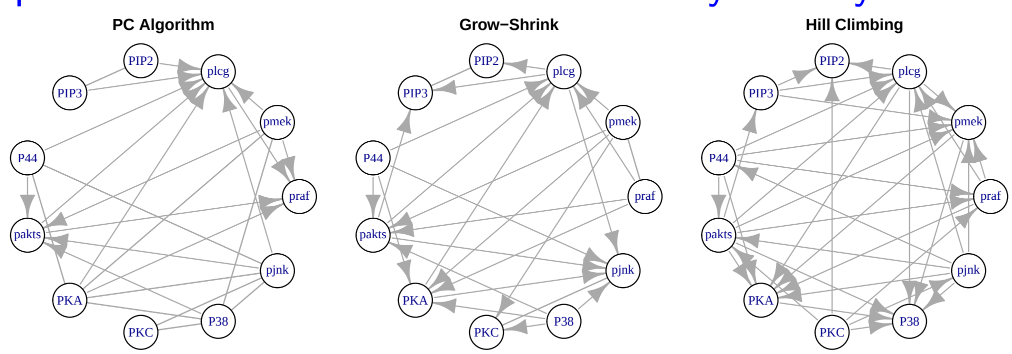

The two graphs are quite different

```
> compare(dag1,dag3)
$tp
[1] 9
$fp
[1] 26
$fn
[1] 17
```

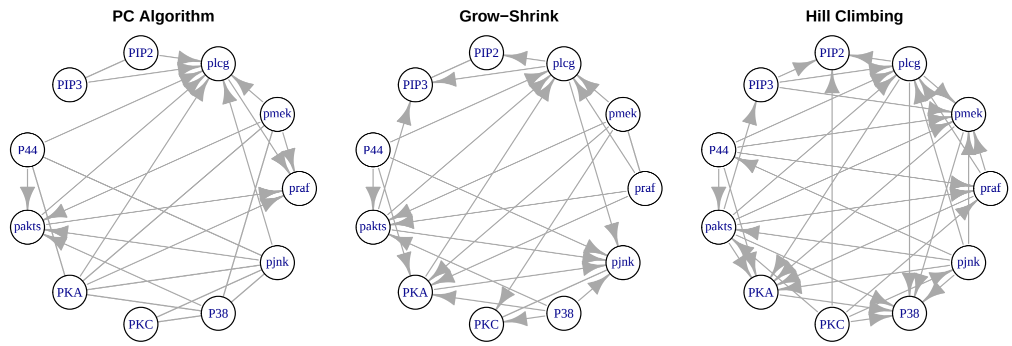




## Comparison of Results for Protein Flow Cytometry Data



# Comparison of Results for Protein Flow Cytometry Data




- The estimated graphs are quite different
- The constrained-based methods seem to have more similarities (at least in terms of structure)
- The estimate from HC has more edges; we can change e.g. the score, but cannot directly control the sparsity





# Penalized Likelihood Estimation of DAGs

- Causal relationships (and probability distributions) on DAGs can be represented using structural equation models

$$X_i = f_i(\mathrm{pa}_i, \gamma_i), \qquad i = 1, \ldots, p$$

- And, for Gaussian random variables, we can write

$$X_i = \sum_{j \in \mathrm{pa}_i} \rho_{ji} X_j + \gamma_i, \qquad i = 1, \ldots, p$$

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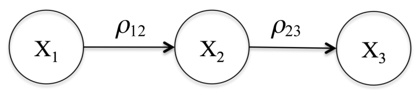






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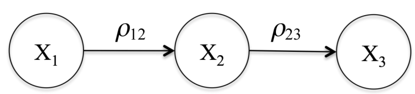

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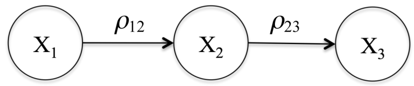
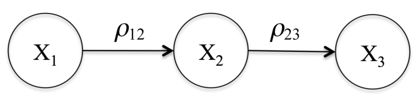



$$
\begin{aligned}
X_1 &= \gamma_1 \\
X_2 &= \rho_{12} X_1 + \gamma_2 = \rho_{12}\gamma_1 + \gamma_2 \\
X_3 &= \rho_{23} X_2 + \gamma_3 = \rho_{23}\rho_{12}\gamma_1 + \rho_{23}\gamma_2 + \gamma_3
\end{aligned}
$$





# Penalized Likelihood Estimation of DAGs

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\end{aligned}
$$

Thus $X = \Lambda\gamma$ where

$$
\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & 1 & 0 \\ \rho_{12}\rho_{23} & \rho_{23} & 1 \end{pmatrix}
$$

## Penalized Likelihood Estimation of DAGs

[1] S & Michailidis (2010)





## Penalized Likelihood Estimation of DAGs

- It turns out that $\Lambda = (I - A)^{-1}$, where $A$ is the weighted adjacency matrix of the DAG[1]
- Thus, for Gaussian random variables, if we know the ordering of the variables (which is a BIG assumption!)

  `after some math...`

  we can estimate the adjacency matrix of DAGs, by minimizing the log-likelihood as a function of $A$:

$$\hat{A} = \arg\min_{A \in \mathcal{A}} \left\{ \mathrm{tr}\left[ (I - A)^{\mathsf{T}} (I - A) S \right] \right\}$$

[1] S & Michailidis (2010)

# Penalized Likelihood Estimation of DAGs

- In high dimensions, we can solve a penalized version of this problem, e.g. by adding a lasso penalty $\lambda \sum_{i<j} |A_{ij}|$
- It turns out that, the problem can be reformulated as $(p-1)$ lasso problems, where we regress each variable, on those appearing earlier in the ordering:

$$\hat{A}_{k,1:k-1} = \arg\min_{\theta \in \mathbb{R}^{k-1}} \left\{ n^{-1} \|X_{1:k-1}\theta - X_{,k}\|_2^2 + \lambda \sum_{j=1}^{k-1} |\theta_j| w_j \right\}$$

- As in `glasso`, $\lambda$ controls the sparsity; $\lambda = \frac{2}{\sqrt{n}} Z_{\alpha/(2p^2)}$ controls a false positive probability at level $\alpha$





# Computational Complexity

- Compared to `pcalg`, this method runs much faster: $\sim np^2$ operations vs $\sim p^q$ ($q$ is the max degree)
- Can be easily implemented in `R` as $p-1$ regressions using `glmnet`. A more general version is available in the `spacejam` package, which also includes estimation for non-Gaussian data

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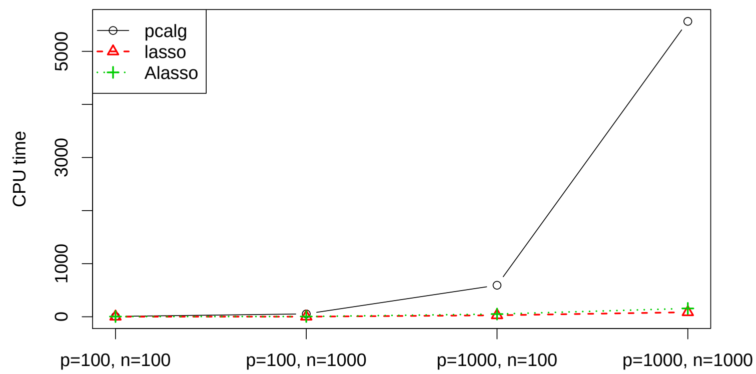


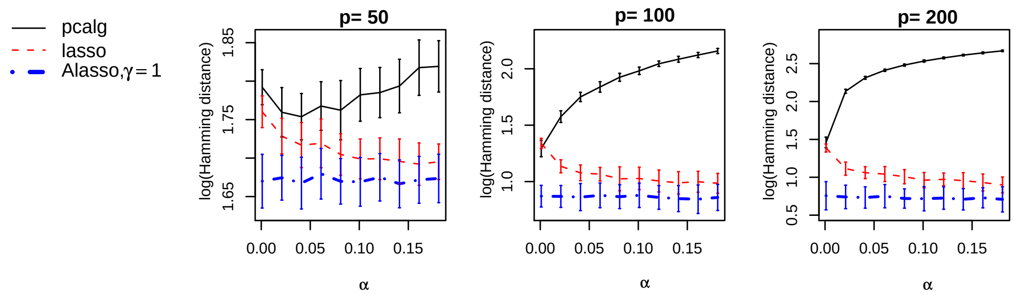





## Simulations

- Settings:
  $p = 50, 100, 200$
  $n = 100$
  Total number of edges in the network $= n$
  100 repetitions
- Performance Criteria
  1. Matthew's Correlation Coefficient (MCC): ranges between $-1$ (worst fit) and 1 (best fit), similar to $F_1$
  2. Structural Hamming Distance (SHD): sum of false positive and false negatives
  3. True positive and false positive rates
- Tuning parameter for both PC-Algorithm and penalized likelihood method based on false positive error $\alpha$

# Gaussian Observations






Introduction
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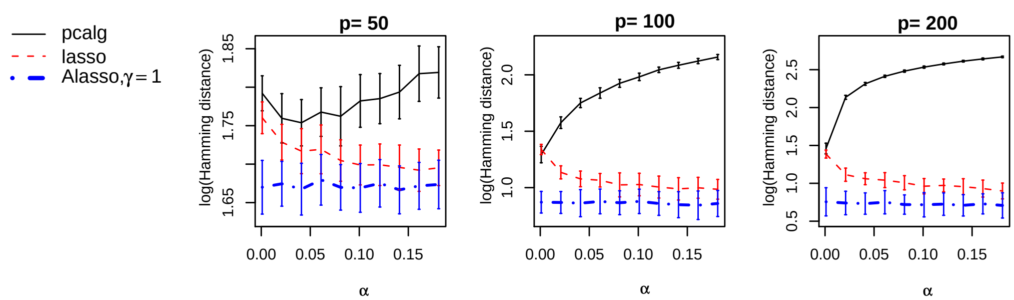


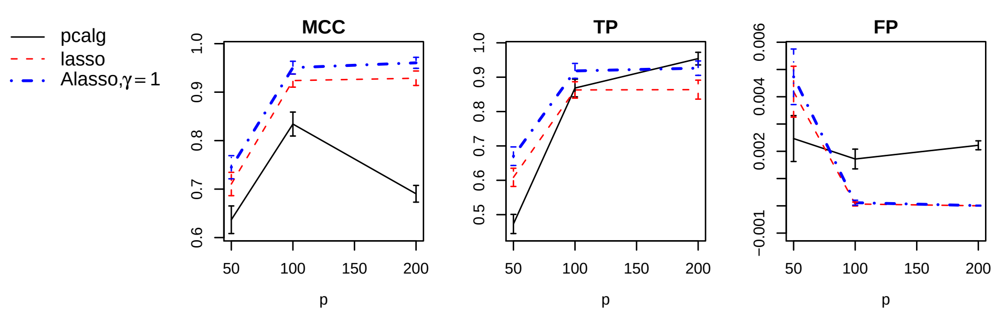


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41

# Regulatory Network of E-Coli

- Regulatory network of E-coli with $p = 49$ genes (7 TFs)
- Want to identify regulatory interactions among TFs and regulated genes





# Regulatory Network of E-Coli

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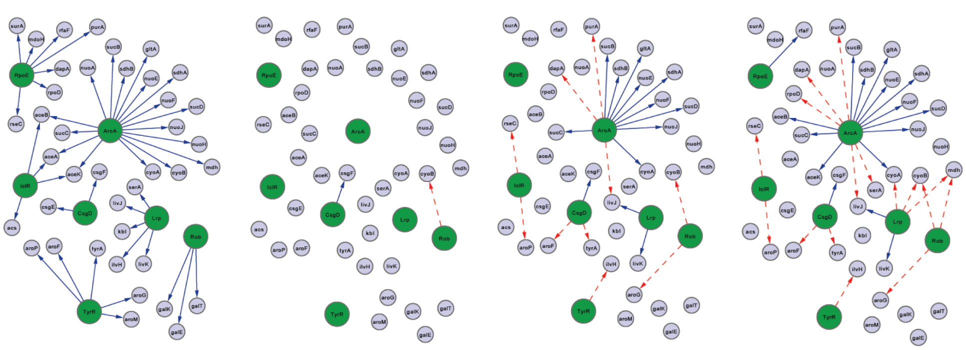
Known Regulatory Network | Pcalg FP=0.004, FN=0.977 MCC=0.08, SHD=43 | Lasso FP=0.042, FN=0.698 MCC=0.34, SHD=40 | Alasso FP=0.068, FN=0.628 MCC=0.35, SHD=43



# DAGs for Time Series Data





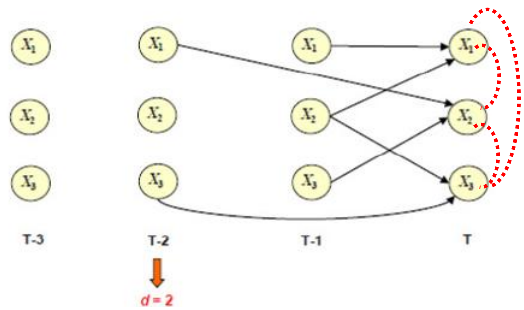
## Time Series Data: A setting where ordering is known

- $p$-dimensional, discrete time, stationary process $X^t = \{X_1^t, \cdots, X_p^t\}$

$$X^t = A_1 X^{t-1} + \cdots + A_d X^{t-d} + \epsilon^t, \quad \epsilon^t \overset{i.i.d}{\sim} N(\mathbf{0}, \Sigma_\epsilon) \quad (1)$$

- $A_1, \ldots, A_d$ : $p \times p$ *transition* matrices (solid, directed edges)
- $\Sigma_\epsilon^{-1}$: contemporaneous dependence (dotted, undirected edges)

# DAGs for Time Series Data




Network *Granger* causality (NGC)





# Network Granger Causality with VARs

- $X_1, \ldots, X_p$: time series for $p$ variables
- $\mathbf{X}^t = (X_1^t, \ldots, X_p^t)'$: realizations at time $t$

## Network Granger Causality with VARs

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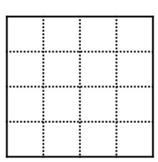

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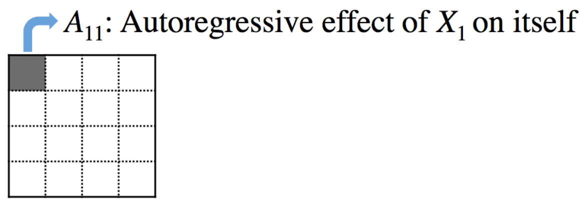






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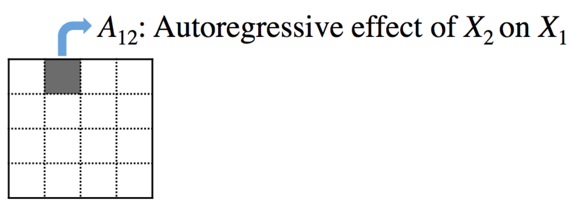

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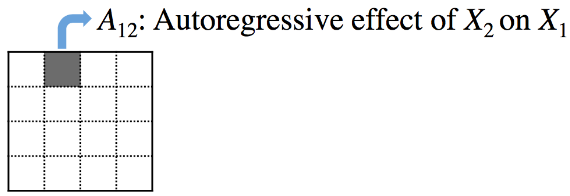



- $X_j$ Granger-causal for $X_i$ if $A_{i,j}^k \neq 0$ for some $k$ ($k = 1, \ldots, d$)





## NGC Estimation

Let $Y$ be the (stacked) vector of current time points; $Z$ be the design matrix based on previous time points; and $\beta$ be

Assuming $A_t$ are sparse, and $d$ is known

- $\ell_1$-penalized least squares ($\ell_1$-LS)

$$\arg\min_{\beta \in \mathbb{R}^{dp^2}} \|Y - Z\beta\|^2 + \lambda \|\beta\|_1$$

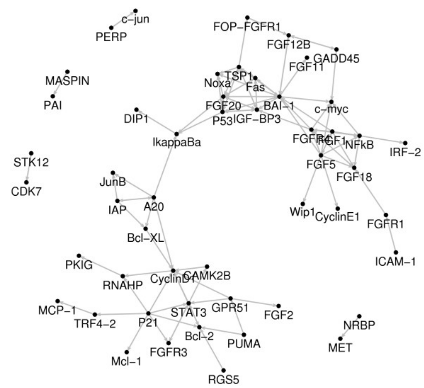
- $\ell_1$-penalized log-likelihood ($\ell_1$-LL) — assuming $\Sigma_\epsilon^{-1}$ is sparse[2]

$$\arg\min_{\beta \in \mathbb{R}^{dp^2}} (Y - Z\beta)' \left(\Sigma_\epsilon^{-1} \otimes I\right)(Y - Z\beta) + \lambda \|\beta\|_1$$

[2] Lin & Michailidis (2017)

# Applications — Functional Genomics

- Identifying regulatory mechanisms using transition patterns in time course expression data
- HeLa gene expression regulatory network (Fujita et al, 2007)

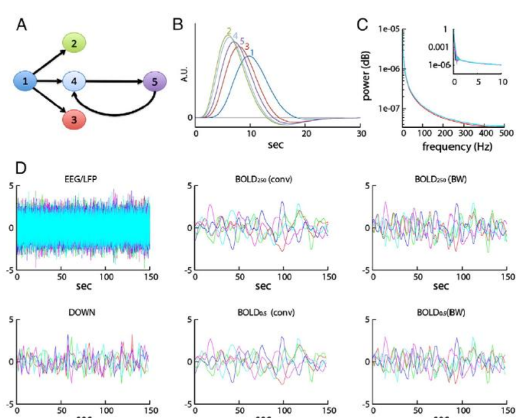







# Applications — Neuroscience

- Connectivity among brain regions from time-course fMRI data
- Connectivity of VAR generative model (Seth et al, 2013)



## Extensions

- Panel VAR Modeling (common in functional genomics and neuroscience)[3]
- Incorporating external information using group lasso penalties, etc[4]
- Dealing with non-statinarity (paucity of long stationary time series — $T$ small)[5]
- Accounting for non-linearity
- ...

[3] S & Michailidis (2010); S, Basu & Michailidis (2012)
[4] Basu, S & Michailidis (2014)
[5] Safikhani & S (2020)

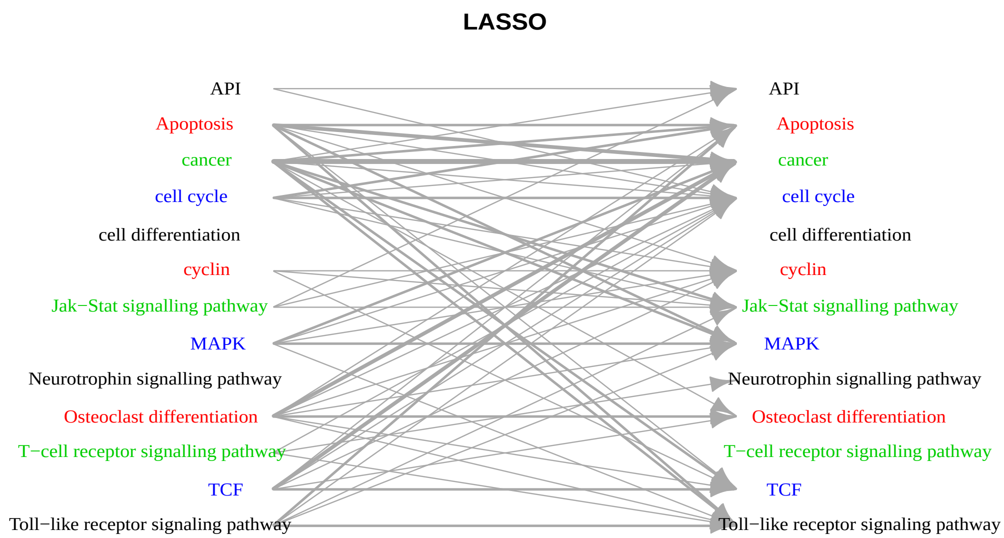




## Example: T-cell Activation Data

- Data from Rangel et al (2004) on T-cell activation — less insight and biological knowledge regarding pathways
- $p = 58$ genes, $n = 44$ samples, and $T = 10$ time points — the first 5 time points (0, 2, 4, 6 and 8 hours) were used on a subset of 38 genes for which pathway information avail
- Goal is to estimate regulatory interactions

# Estimated Network Structure





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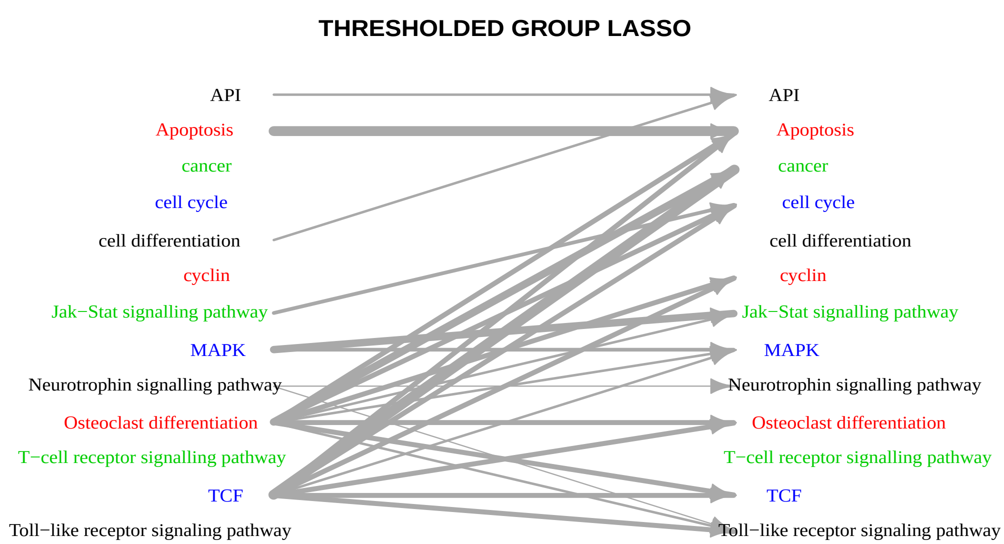

# Summary

- Estimation of DAGs from observational data is both conceptually and computationally difficult
- Constraint-based & search-based algorithms — slow in high dim
- May not be able to distinguish DAGs from observational data (Markov equivalence)
- Efficient penalized likelihood methods can estimate DAGs if the ordering is known
- Important case is time series data, but Granger causality $\neq$ causality![6]
- Efficient implementations in R available for most methods

[6] S & Fox (2022)