Mod 12 Self-Study Material 1: A Brief Review of Differentiation and Integration

The **derivative** of a function $y = f(x)$, evaluated at a point $x = x_0$, provides the instantaneous rate of change of $f(x)$ at $x = x_0$, that is, the slope of the tangent line to the curve $y = f(x)$ at $x = x_0$. It is denoted in many ways - for example, as $y'(x_0)$, $f'(x_0)$, $\frac{dy}{dx}(x_0)$, $\frac{df}{dx}(x_0)$, or $\frac{d}{dx}f(x)|_{x=x_0}$.

Mathematical definition:

$$f'(x) \ = \ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Basic properties and rules:
Let $f(x)$ and $g(x)$ be two functions, and let c be any constant.

- (Constant multiplier rule) $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$

- (Derivative of a constant is zero) $\frac{d}{dx}(c) = 0$

- (Addition rule) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

- (Product rule) $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

- (Quotient rule) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

- (Chain rule) $\frac{d}{dx}(f(g(x))) = \frac{d}{du}f(u)|_{u=g(x)}\frac{d}{dx}g(x)$

- (Power rule) $\frac{d}{dx}(x^c) = cx^{c-1}$

- (Monotonicity) If $f'(x) \geq 0$ (or $f'(x) \leq 0$) on an interval I, then $f(x)$ is non-decreasing (or non-increasing) on I. If $f'(x) > 0$ (or $f'(x) < 0$) on an interval I, then $f(x)$ is increasing (or decreasing) on I.

Derivatives of exponential and logarithmic functions:
Let a be a positive constant.

$\frac{d}{dx}(a^x) = \ln(a)\ a^x$ and so, setting $a = e$, we obtain $\frac{d}{dx}(e^x) = e^x$.

$\frac{d}{dx}(\log_a x) = (\ln(a)\ x)^{-1}$ and so, setting $a = e$, we obtain $\frac{d}{dx}(\ln x) = 1/x$.

Recall the following facts:
1) $\ln x = \log_e x$, 2) $\log_a b = c$ is equivalent to $a^c = b$, 3) $\log_a b^c = c \log_a b$, 4) $\log_a (bc) = \log_a b + \log_a c$ and 5) $\log_a (b/c) = \log_a b - \log_a c$.

An **antiderivative** of a function $f(x)$ is any function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$. Note that if $F_0(x)$ is such a function, then so is $F_c(x) = F_0(x) + c$ for any real number c. The class of all antiderivatives of a function $f(x)$ is referred to as the (indefinite) integral of $f(x)$. It is denoted by $\int f(x)dx$.

The (definite) integral of $f(x)$ over $[a,b]$ is defined as $F(b)-F(a)$ (often denoted as $F(x)|_{x=a}^{b}$), where $F(x)$ is any antiderivative of $f(x)$, and it is denoted by $\int_a^b f(x)dx$. Note that the definition yields the same result no matter the antiderivative chosen.

As with the derivative, the integral also is interpretable geometrically. Indeed, the definite integral of $f(x)$ over $[a,b]$ gives the (signed) area of the region bounded by the $x-axis$ and the curve $y=f(x)$, between $x=a$ and $x=b$.

Basic properties and rules:
Let $f(x)$ and $g(x)$ be two functions, and let a,b and c be constants. Let $F(x)$ be an antiderivative of $f(x)$.

- (Constant multiplier rule) $\int cf(x)dx = c\int f(x)dx$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

- (Addition rule)
$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx$ and $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

- (Null integral) $\int_a^a f(x)dx = 0$

- (Reversal of integral) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

- (Partition of integral) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

- (Positivity) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

- (Dominance) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

- (Power rule) $\int x^a dx = x^{a+1}/(a+1)$

Antiderivatives of exponential functions:
Let a be a positive constant.

$\int a^x dx = a^x/\ln(a)$ and so, setting $a=e$, we obtain $\int e^x dx = e^x$.

Integration by substitution:
Method that ravels back together the result of a chain rule differentiation - i.e. the integration analog of the chain rule.

Suppose $f(x)$ and $g(x)$ are two functions, and that $F(x)$ is an antiderivative of $f(x)$. Then, we have that $\int f(g(x))g'(x)dx = F(g(x))$. Thus, if your integrand is a simple function to integrate but its argument is a more complicated function AND the derivative of this more complicated function is present (up to a constant multiple), then integration by substitution may be applicable.

A useful extra - the Gamma function:

We define the Gamma function as $\Gamma(\alpha) = \int_{x=0}^{\infty} e^{-x}x^{\alpha-1}dx$, for $\alpha > 0$. This function is also defined on the negative integers and at zero, but we do not require this additional fact. The Gamma function satisfies a fundamental property, namely that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. It follows then that for any integer $n \geq 1$, we have $\Gamma(n) = (n-1)! = (n-1)(n-2)...1$. This equation turns out very useful when calculating moments of several distributions (such as the exponential and Gamma distributions).